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On Numerical Solutions of Stefan Problem II.
Unique Existence of Numerical Solution.

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§1. Introduction

Previously we presented a numerical method for unidimensional Stefan problem[1]. In our method, the system of differential and integral equations with penalty term is discretized and solved by iteration. The accuracy of numerical results is fairly good.

In this paper, we slightly modify the penalty function and prove the unique existence of the numerical solution.

§2. Notations and Problem

2.1 Notations

Notations used in this paper are almost the same as those in the previous paper[1]. We list up new notations and important ones in the following;

1° $\lambda = k/h^2$,

2° $T = Nk$,

3° $X = Mh$,

4° $C = \max(\max_n f_n, \max_m \phi_m)$,

5° $A = \max(C/b, D)$, $b > 0$,

6° $m_n = [s_n/h]$,

7° $\rho_n = s_n/h - [s_n/h]$,

8° $\varepsilon_{m,n} = 1/(1 + kK\chi_{m,n})$,

9° $\delta u_{m,n} = u_{m-1,n} - 2u_{m,n} + u_{m+1,n}$,

10° $Pu_{m,n} = (I + \lambda\delta)u_{m,n}$,

11° $\|u\|_n = \max_{0 \leqq m \leqq M} |u_{m,n}|$.

2.2 Problem

Our problem is to seek two functions $u_{m,n}$ and s_n which satisfy the following difference equations;

(2.1) $$\frac{u_{m,n} - u_{m,n-1}}{k} = \frac{1}{h^2}\delta u_{m,n-1} - K\chi_{m,n}u_{m,n} ,$$

$$1 \leqq m \leqq M-1 , \quad 1 \leqq n \leqq N ,$$

(2.2) $u_{0,n} = f_n \geqq 0$, $0 \leqq n \leqq N$,

(2.3) $u_{M,n} = 0$, $0 \leqq n \leqq N$,

(2.4) $$u_{m,0} = \begin{cases} \phi_m , & 1 \leqq m \leqq [b/h] , \quad 0 \leqq \phi_m \leqq D([b/h]+1-m)h , \\ 0 , & [b/h]+1 \leqq m \leqq M , \end{cases}$$

$$(2.5)\qquad s_n = \sqrt{F_n - 2h^2 \sum_{m=1}^{M-1} m u_{m,n}}\ , \quad 0 \leq n \leq N\ ,$$

where

$$F_n = b^2 + 2k \sum_{i=0}^{n-1} f_i + 2h^2 \sum_{m=1}^{[b/h]} m\phi_m$$

$$(2.6)\qquad \chi_{m,n} = \begin{cases} 0 & , \quad 1 \leq m \leq m_n - 1 \ , \quad 0 \leq n \leq N\ , \\ \dfrac{1-\rho_n}{1+\rho_n kK} & , \quad m = m_n \ , \quad 0 \leq n \leq N\ , \\ 1 & , \quad m_n + 1 \leq m \leq M\ , \quad 0 \leq n \leq N\ . \end{cases}$$

2.3 Penalty function $\chi_{m,n}$

Rewriting (2.1) we get

$$(2.7)\qquad u_{m,n} = \varepsilon_{m,n} P u_{m,n-1}\ ,$$

where

$$(2.8)\qquad \varepsilon_{m,n} = \begin{cases} 1 & , \quad 1 \leq m \leq m_n - 1 \ , \quad 0 \leq n \leq N\ , \\ \dfrac{1+\rho_n kK}{1+kK} & , \quad m = m_n \ , \quad 0 \leq n \leq N\ , \\ \dfrac{1}{1+kK} & , \quad m_n + 1 \leq m \leq M\ , \quad 0 \leq n \leq N\ . \end{cases}$$

As shown in (2.8), the value of $\varepsilon_{m,n}$ (i.e., $\chi_{m,n}$) at $m=m_n$ differs from that in the previous scheme. The reason of this modification is as follows. In the previous scheme, only the integer part of s_n/h was concerned to determine $\varepsilon_{m,n}$ (i.e., $\chi_{m,n}$), therefore continuity of $u_{m,n}$ to s_n was lost. To take back the continuity, we consider the fractional part ρ_n of s_n/h. That is, we define the amplification factor $\varepsilon_{m,n}$ at $m=m_n$ by the linear interpolation

$$(2.9) \qquad \varepsilon_{m_n,n} = \rho_n \cdot 1 + (1-\rho_n)\cdot \frac{1}{1+kK} = \frac{1+\rho_n kK}{1+kK} .$$

From the relation

$$\varepsilon_{m,n} = \frac{1}{1+kK\chi_{m,n}} ,$$

the value of the penalty function $\chi_{m,n}$ at $m=m_n$ is written as

$$\chi_{m_n,n} = \frac{1-\rho_n}{1+\rho_n kK} .$$

§3. Algorithm and Result

Algorithm to solve the equations (2.1)~(2.6) is the same as in [1]. That is, we choose the 0-th approximation of s_n as

$$(3.1) \qquad s_n^{(0)} = \sqrt{F_n} , \quad 0 \leq n \leq N .$$

We solve the difference equation (2.1) using the penalty function $\chi_{m,n}^{(0)}$ determined by $s_n^{(0)}$ and we denote the solution by $u_{m,n}^{(0)}$. Substituting this solution into (2.5) we get the 1st approximation of s_n as

$$(3.2) \qquad s_n^{(1)} = \sqrt{F_n - 2h^2 \sum_{m=1}^{M-1} m u_{m,n}^{(0)}} , \quad 0 \leq n \leq N .$$

Repeating the similar procedure we calculate $u_{m,n}^{(\ell)}$ and $s_n^{(\ell)}$.

The object of this paper is to show that $u_{m,n}^{(\ell)}$ and $s_n^{(\ell)}$ converge to the unique solution of (2.1)~(2.6) as $\ell \to \infty$.

Theorem. Suppose that

$$(1) \qquad 0 < \lambda \leq \frac{1}{2} ,$$

(2) $K = 1/k^2$,

(3) $0 < C_2 < 1$, where $C_2 = \frac{X}{b}(A + C\lambda h)h$

then there exists uniquely the solution of (2.1)∼(2.6).

§4. Preliminaries

As a preparation for the proof of the Theorem, we state some definitions and lemmas.

Definition 1. Let $s = \{s_0, s_1, \cdots, s_N\}$. s is said to have property (M) if it satisfies the relation

$$b \leq s_0 \leq s_1 \leq \cdots \leq s_N \leq X .$$

Definition 2. Let the solution of (2.1)∼(2.4) and (2.6) for a given s be $u_{m,n}$. We define

$$\Phi(s) = \{\Phi_0(s), \Phi_1(s), \cdots, \Phi_N(s)\}$$

by

$$\Phi_n(s) = \sqrt{F_n - 2h^2 \sum_{m=1}^{M-1} m u_{m,n}} , \quad 0 \leq n \leq N . \tag{4.1}$$

From the argument in §2.3, we see that Φ is continuous with respect to s. If s is the true boundary, then it holds

$$s = \Phi(s) \quad \text{i.e.,} \quad s_n = \Phi_n(s) , \quad 0 \leq n \leq N .$$

Lemma 1. (Monotone dependence of $u_{m,n}$ on s)

Let solution of (2.1)∼(2.4) and (2.6) for given s' and s'' be $u'_{m,n}$ and $u''_{m,n}$ respectively. Suppose

$$0 \leqq s'_n \leqq s''_n \leqq X \quad , \quad 0 \leqq n \leqq N \quad ,$$

and $\quad 0 < \lambda \leqq 1/2 \quad ,$

then it holds

$$0 \leqq u'_{m,n} \leqq u''_{m,n} \quad , \quad 0 \leqq m \leqq M \quad , \quad 0 \leqq n \leqq N \quad .$$

Proof. It is proved by the same argument as in [1].

Lemma 2. Suppose s have property (M), then $\Phi(s)$ also has property (M).

Proof. By the definition of $\Phi(s)$ we have

$$\{\Phi_{n+1}(s)\}^2 = F_{n+1} - 2h^2 \sum_{m=1}^{M-1} mu_{m,n+1} \quad ,$$

$$\{\Phi_n(s)\}^2 = F_n - 2h^2 \sum_{m=1}^{M-1} mu_{m,n} \quad .$$

Therefore we get

$$\{\Phi_{n+1}(s)\}^2 - \{\Phi_n(s)\}^2 = 2kf_n - 2h^2 \sum_{m=1}^{M-1} m(\varepsilon_{m,n+1}P - I)u_{m,n}$$

$$\geqq 2kf_n - 2h^2 \sum_{m=1}^{M-1} m(P - I)u_{m,n}$$

$$= 2kf_n - 2h^2 \sum_{m=1}^{M-1} m\lambda\delta u_{m,n}$$

$$= 2k(M - 1)u_{M-1,n} \geqq 0 \quad . \quad \text{(Lemma 1)}$$

It is obviously seen that

$$\Phi_0(s) = \sqrt{F_0 - 2h^2 \sum_{m=1}^{M-1} m\phi_m} = b \quad ,$$

$$\Phi_N(s) = \sqrt{F_N - 2h^2 \sum_{m=1}^{M-1} mu_{m,N}} \leq \sqrt{F_N} \leq X .$$

Q.E.D.

Lemma 3. (The estimation of $u_{m,n}$ for $m_n \leq m \leq M$)

Suppose a given s have property (M) and $K=1/k^2$, then the solution $u_{m,n}$ of (2.1)∿(2.4) and (2.6) is estimated as follows;

(4.2) $\quad 0 \leq Pu_{m,n-1} \leq C_1 h \ , \ m_n \leq m \leq M \ , \ 1 \leq n \leq N \ ,$

where

$$C_1 = A + C\lambda h .$$

Proof. We introduce a function $v_{m,n}$ which is a solution of the same equations for $u_{m,n}$ except that the boundary condition (2.3) is replaced by

(4.3) $\quad v_{m,n} = 0 \ , \ m_n+1 \leq m \leq M \ , \ 0 \leq n \leq N \ .$

First we consider the estimation of $u_{m,n}$ for the case $m=m_n$.

(4.4) $\quad Pu_{m_n,n-1} \leq |Pu_{m_n,n-1} - Pv_{m_n,n-1}| + Pv_{m_n,n-1} .$

Let us estimate the first term in the right hand side of (4.4). From the initial and the boundary condition (2.2) and (2.4), we get

$$u_{0,n} = v_{0,n} ,$$

$$u_{m,0} = v_{m,0} .$$

From (2.7) and (4.3) follows

$$0 \leqq u_{m_n+1,n} - v_{m_n+1,n} \leqq \frac{C}{1 + kK} .$$

By the use of Maximum principle, there holds for any m and n $(1 \leqq m \leqq m_n,\ 0 \leqq n \leqq N)$

$$|u_{m,n} - v_{m,n}| \leqq \frac{C}{1 + kK} .$$

In the above inequality we put $m=m_n$ and $n=n-1$ and operate P on both sides. Then we obtain the estimation ;

$$(4.5) \quad |Pu_{m_n,n-1} - Pv_{m_n,n-1}| \leqq \frac{C}{1 + kK} .$$

Next we estimate the second term in the right hand side of (4.4). If we use the difference version of Lemma 1 in [2], there holds the following estimation, (see Appendix I)

$$(4.6) \quad v_{m_n,n} \leqq Ah .$$

Considering the fact that s has property (M), we see

$$v_{m_n+1,n-1} = 0 ,$$

$$0 \leqq v_{m_n,n-1} \leqq Ah ,$$

$$0 \leqq v_{m_n-1,n-1} \leqq 2Ah .$$

Thus it follows

$$(4.7) \quad Pv_{m_n,n-1} \leqq Ah .$$

From (4.5) and (4.7) we have

$$0 \leqq Pu_{m_n,n-1} \leqq \frac{C}{1 + kK} + Ah \leqq (A + C\lambda h)h = C_1 h \ ,$$

where we used the assumption $K=1/k^2$.

Next we deal with $u_{m,n}$ in the domain $m_n+1 \leqq m \leqq M,\ 0 \leqq n \leqq N$. By the initial and the boundary conditions we see

$$u_{M,n} = 0 \ ,$$

$$u_{m,0} = 0 \ .$$

The result for the case $m=m_n$ leads

$$u_{m_n,n} = \varepsilon_{m_n,n} Pu_{m_n,n-1} \leqq Pu_{m_n,n-1} \leqq C_1 h \ .$$

By virture of Maximum principle, we get the estimation of $u_{m,n}$ for any m and n in this domain

$$0 \leqq u_{m,n} \leqq C_1 h \ .$$

In the above inequality we put $n=n-1$ and operate P on both sides, then we obtain

$$0 \leqq Pu_{m,n-1} \leqq C_1 h \qquad (m_n+1 \leqq m \leqq M,\ 1 \leqq n \leqq N) \ .$$

Q.E.D.

Lemma 4. Suppose s' and s'' have property (M), then it holds

$$(4.9) \qquad \|u' - u''\|_n \leqq C_1 \sum_{i=1}^{n} |s'_i - s''_i| \ .$$

Proof. Even if we suppose $s'_n \leqq s''_n$, we do not lose generality. From (2.7) we have

$$u'_{m,n} - u''_{m,n} = \varepsilon'_{m,n} P u'_{m,n-1} - \varepsilon''_{m,n} P u''_{m,n-1}$$

$$= (\varepsilon'_{m,n} - \varepsilon''_{m,n}) P u'_{m,n-1} + \varepsilon''_{m,n} P (u'_{m,n-1} - u''_{m,n-1}) .$$

Therefore it holds

$$\|u' - u''\|_n \leqq \max_{0 \leqq m \leqq M} (\varepsilon''_{m,n} - \varepsilon'_{m,n}) \times \max_{m'_n \leqq m \leqq m''_n} P u'_{m,n-1} + \|u' - u''\|_{n-1} .$$

Note that $\varepsilon''_{m,n} - \varepsilon'_{m,n} = 0$ for $0 \leqq m \leqq m'_n - 1$ and $m''_n + 1 \leqq m \leqq M$. By the definition there holds

$$(4.10) \qquad (s''_n - s'_n)/h = m''_n - m'_n + \rho''_n - \rho'_n .$$

We consider the following three cases.

Case 1. $m''_n = m'_n$,

$$\max_m (\varepsilon''_{m,n} - \varepsilon'_{m,n}) = \frac{(\rho''_n - \rho'_n)kK}{1 + kK} < \frac{s''_n - s'_n}{h} .$$

Case 2. $m''_n = m'_n + 1$,

$$\max_m (\varepsilon''_{m,n} - \varepsilon'_{m,n}) = \max \left(\frac{(1 - \rho'_n)kK}{1 + kK} , \frac{\rho''_n kK}{1 + kK} \right) < \frac{s''_n - s'_n}{h} .$$

In fact it is obvious from the relation

$$(s''_n - s'_n)/h = (1 - \rho'_n) + \rho''_n ,$$

$$0 \leqq \rho'_n < 1 , \quad 0 \leqq \rho''_n < 1 .$$

Case 3. $m''_n \geqq m'_n + 2$,

$$\max_m (\varepsilon''_{m,n} - \varepsilon'_{m,n}) = \frac{kK}{1 + kK} < \frac{s''_n - s'_n}{h} .$$

In fact it is derived from the following relation

$$(s''_n - s'_n)/h = (m''_n - m'_n) + (\rho''_n - \rho'_n) > 1 .$$

Where we used the fact

$$m''_n - m'_n \geqq 2 , \quad \rho''_n - \rho'_n > -1 .$$

From the above result and Lemma 3, we get

$$(4.11) \qquad \|u' - u''\|_n \leqq C_1 |s'_n - s''_n| + \|u' - u''\|_{n-1} .$$

Considering the initial condition $\|u' - u''\|_0 = 0$ and the recurrence formula (4.11), we have (4.9).

Q.E.D.

Lemma 5. Suppose s' and s'' have property (M), then there holds

$$(4.12) \qquad |\Phi_n(s') - \Phi_n(s'')| \leqq C_2 |s'_n - s''_n| + C_1 C_3 \sum_{i=1}^{n-1} |s'_i - s''_i| ,$$

where

$$C_2 = \frac{X}{b} C_1 h ,$$

$$C_3 = \frac{X^2}{2b} .$$

Proof. By the definition

$$(4.13)\quad \{\Phi_n(s')\}^2 - \{\Phi_n(s'')\}^2$$

$$= 2h^2 \sum_{m=1}^{M-1} m(u''_{m,n} - u'_{m,n})$$

$$= 2h^2 \sum_{m=1}^{M-1} m(u''_{m,n} - u'''_{m,n}) + 2h^2 \sum_{m=1}^{M-1} m(u'''_{m,n} - u'_{m,n}) ,$$

where

$$u'''_{m,n} = \varepsilon''_{m,n} P u'_{m,n-1} .$$

The first term in the right hand side of (4.13) is estimated as follows;

$$2h^2 \sum_m m\varepsilon''_{m,n} P(u''_{m,n-1} - u'_{m,n-1}) \leqq X^2 \|u'' - u'\|_{n-1} .$$

Next we estimate the second term. Generality is not lost even if we assume $s'_n \leqq s''_n$. In the case $m''_n \geqq m'_n+2$, we have

$$2h^2 \sum_m m(\varepsilon''_{m,n} - \varepsilon'_{m,n}) P u'_{m,n-1}$$

$$\leqq 2hXC_1 h \sum_{m=m'_n}^{m''_n} (\varepsilon''_{m,n} - \varepsilon'_{m,n}) \qquad \text{(Lemma 3)}$$

$$\leqq 2XC_1 h^2 \{(1 - \rho'_n) + \sum_{m=m'_n+1}^{m''_n-1} 1 + \rho''_n\}$$

$$= 2XC_1 h^2 (m''_n - m'_n + \rho''_n - \rho'_n)$$

$$= 2XC_1 h (s''_n - s'_n) .$$

The same result is obtained in the cases $m''_n=m'_n$ and $m''_n=m'_n+1$. From the estimations mentioned above, we get

$$(4.14)\quad \{\Phi_n(s')\}^2 - \{\Phi_n(s'')\}^2 \leqq 2XC_1 h|s'_n - s''_n| + X^2\|u'' - u'\|_{n-1} ,$$

which follows

$$(4.15) \qquad |\Phi_n(s') - \Phi_n(s'')| \leqq C_2\, |s'_n - s''_n| + C_3\, \|u'' - u'\|_{n-1} ,$$

where we used the following relation obtained by Lemma 2,

$$\Phi_n(s') + \Phi_n(s'') \geqslant 2b \quad , \quad 0 \leqq n \leqq N \quad .$$

From (4.15) and Lemma 4, we have (4.12).

Q.E.D.

§5. Proof of the Theorem

5.1 $s^{(\ell)}$ $(\ell \geqq 0)$ defined in §3 has property (M).

In fact by the definition

$$s_n^{(0)} = \sqrt{b^2 + 2k \sum_{i=0}^{n-1} f_n + 2h^2 \sum_{m=1}^{[b/h]} m\phi_m} \quad .$$

Therefore

$$s_0^{(0)} \geqq b \ .$$

Since $f_n \geqq 0$, it is seen that

$$s_n^{(0)} \geqq s_{n-1}^{(0)} ,$$

and it is obvious that

$$s_N^{(0)} = \sqrt{F_N} \leqq X \ .$$

Thus $s^{(0)}$ has property (M). Next from the relation

$$s^{(\ell)} = \Phi(s^{(\ell-1)})$$

and Lemma 2, it is seen that $s^{(\ell)}$ $(\ell \geq 1)$ has property (M).

5.2 It is obvious that $s_0^{(\ell)} = b$ $(\ell \geq 1)$ by the definition in §3. Genarally for $n \geq 1$, there holds by Lemma 5,

$$(5.1) \qquad |s_n^{(\ell+1)} - s_n^{(\ell)}| \leq C_2 |s_n^{(\ell)} - s_n^{(\ell-1)}| + C_1 C_3 \sum_{i=1}^{n-1} |s_i^{(\ell)} - s_i^{(\ell-1)}| \; .$$

If we put

$$(5.2) \qquad |s_n^{(\ell+1)} - s_n^{(\ell)}| = Q^{(n)}(\ell) C_2^{\ell} \; ,$$

then we obtain

$$(5.3) \qquad Q^{(n)}(\ell) \leq Q^{(n)}(\ell-1) + \frac{C_1 C_3}{C_2} \sum_{i=1}^{n-1} Q^{(i)}(\ell-1) \quad .$$

Using the fact that it holds for any n

$$(5.4) \qquad Q^{(n)}(0) = |s_n^{(1)} - s_n^{(0)}| \leq X - b \quad ,$$

we get the general expression as follows; (see Appendix II)

$$(5.5) \qquad Q^{(n)}(\ell) \leq (X - b) \sum_{p=0}^{n-1} \left(\frac{C_1 C_3}{C_2}\right)^p \binom{n-1}{p} \binom{\ell}{p} \quad .$$

That is, $Q^{(n)}(\ell)$ is estimated by a polynomial of $(n-1)$-th order of ℓ. Therefore we have from (5.2)

$$(5.6)\qquad \max_{0\leq n\leq N} |s_n^{(\ell+1)} - s_n^{(\ell)}| \leq [\max_{0\leq n\leq N} Q^{(n)}(\ell)]\cdot C_2^{\ell}$$

$$\leq O(\ell^{N-1})\ C_2^{\ell}\ .$$

By the assumption $0 < C_2 < 1$, the right hand side of (5.6) converges to 0 as $\ell\to\infty$. That is, $s^{(\ell)}$ has the limit $\hat{s}$. It is evident that $\hat{s}$ has also property (M). By the relation

$$s^{(\ell+1)} = \Phi(s^{(\ell)})\quad ,$$

and the continuity of Φ, there holds as $\ell\to\infty$

$$\hat{s} = \Phi(\hat{s})\quad ,$$

that is, $\hat{s}$ is the boundary in search. If we denote the corresponding solution by $\hat{u}_{m,n}$, $(\hat{s}\ ,\ \hat{u}_{m,n})$ is the solution of the difference system (2.1)~(2.6).

5.3 Uniqueness of Solution

Suppose $(s'\ ,\ u')$ and $(s''\ ,\ u'')$ be solutions of the difference system (2.1)~(2.6). We assume that s'_n and s''_n are equal for $0 \leq n \leq n_0-1$ and differ for the first time at $n = n_0$. We do not lose generality even if we assume $s'_{n_0} < s''_{n_0}$. By Lemma 1

$$(5.7)\qquad u'_{m,n_0} < u''_{m,n_0}\quad ,\quad 0 \leq m \leq M\quad .$$

From (2.5)

$$s'_{n_0} = \sqrt{F_{n_0} - 2h^2\sum_{m=1}^{M-1} mu'_{m,n_0}}$$

and

$$s''_{n_0} = \sqrt{F_{n_0} - 2h^2 \sum_{m=1}^{M-1} mu''_{m,n_0}} .$$

Using (5.7) we get

$$s'_{n_0} > s''_{n_0} .$$

This contradicts the assumption. Therefore it holds

$$s'_{n_0} = s''_{n_0} .$$

It is clear that $s'_0 = s''_0$. By mathematical induction we see $s'_n = s''_n$ for any n $(0 \leq n \leq N)$. Obviously the corresponding solutions $u'_{m,n}$ and $u''_{m,n}$ are coincident.

References

[1] H. Kawarada and M. Natori, "On Numerical Solutions of Stefan Problem.I" , Memoirs of Numerical Mathematics, No.1 (1974) 43-54

[2] J. R. Cannon and C. D. Hill, "Existence, Uniqueness, Stability, and Monotone Dependence in a Stefan Problem for the Heat Equation" , J. Math. and Mech. 17 (1967) 1-19

Appendix I. Proof of (4.6)

$v_{m,n}$ satisfies the following difference system;

$$v_{m,n} = \varepsilon_{m,n} P v_{m,n-1} \;, \quad 1 \leq m \leq m_n \;, \quad 1 \leq n \leq N \;,$$

$$v_{0,n} = f_n \;, \quad 0 \leq n \leq N \;,$$

$$v_{m,0} = \phi_m \;, \quad 1 \leq m \leq [b/h] \;,$$

$$v_{m_n+1,n} = 0 \;, \quad 0 \leq n \leq N \;.$$

Define for each n_0, $0 \leq n_0 \leq N$

$$W_{m,n} = A([s_{n_0}/h] + 1 - m)h \;, \; 0 \leq m \leq [s_{n_0}/h] + 1 \;,$$

$$0 \leq n \leq n_0 \;.$$

Observe that $W_{m,n}$ satisfies the difference equation

$$W_{m,n} = PW_{m,n-1} \;, \quad 1 \leq m \leq m_n \;, \quad 1 \leq n \leq n_0 \;,$$

with the initial and boundary conditions

$$W_{0,n} = A([s_{n_0}/h] + 1) \geq A([\, b/h] + 1)h$$

$$\geq Ab \geq C \geq f_n \;,$$

$$W_{m,0} = A([s_{n_0}/h] + 1 - m)h \geq A([b/h] + 1 - m)h$$

$$\geq D([b/h] + 1 - m)h \geq \phi_m \;.$$

and

$$W_{m_n+1,n} = A([s_{n_0}/h] - m_n)h \geq 0 \quad .$$

Comparing $W_{m,n}$ with $v_{m,n}$ in $0 \leq m \leq m_{n+1}$, $0 \leq n \leq n_0$, there holds by Maximum Principle,

$$\text{(I.1)} \qquad 0 \leq v_{m,n} \leq W_{m,n} \quad .$$

Note that $0 < \varepsilon_{m,n} \leq 1$.

Put $m = [s_{n_0}/h]$, $n = n_0$ in (I.1) , then we get

$$\text{(I.2)} \qquad v_{[s_{n_0}/h],n_0} \leq Ah \quad ,$$

for each n_0 , $0 \leq n_0 \leq N$.

Replace n_0 by n in (I.2) , we obtain (4.6) . Q.E.D.

Appendix II. Proof of (5.5)

As a preparation for the proof of (5.5), we state the following formula on binomial coefficients,

$$\text{(II.1)} \qquad \binom{n}{k} = \sum_{i=0}^{n-1} \binom{i}{k-1}$$

where we define that

$$\binom{i}{j} = 0 \ , \ \text{for } i < j \ .$$

This formula is derived from a well-known recurrence relation :

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k} \ .$$

We shall prove (5.5) by induction.

(1st step) When n=1, (5.5) holds evidently.

(2nd step) Suppose there holds for any $i \leqq n-1$

$$\text{(II.2)} \qquad Q^{(i)}(\ell) \leqq (X - b) \sum_{p=0}^{i-1} \binom{i-1}{p} C_4^p \binom{\ell}{p}$$

where

$$C_4 = \frac{C_1 C_3}{C_2} \ .$$

Then it is seen by (5.3)

$$\begin{aligned} Q^{(n)}(\ell) &\leqq Q^{(n)}(\ell-1) + C_4 \sum_{i=1}^{n-1} Q^{(i)}(\ell-1) \\ &\leqq Q^{(n)}(\ell-1) + (X - b) C_4 \sum_{i=1}^{n-1} \sum_{p=0}^{i-1} \binom{i-1}{p} C_4^p \binom{\ell-1}{p} \\ &= Q^{(n)}(\ell-1) + (X - b) C_4 \sum_{p=0}^{n-2} C_4^p \binom{\ell-1}{p} \sum_{i=p+1}^{n-1} \binom{i-1}{p} \end{aligned}$$

o

$$= Q^{(n)}(\ell-1) + (X-b) \sum_{p=0}^{n-2} \binom{n-1}{p+1} C_4^{p+1} \binom{\ell-1}{p} .$$

Repeating use of the above recurrence relation with respect to ℓ leads

$$\begin{aligned} Q^{(n)}(\ell) &\leqq Q^{(n)}(0) + (X-b) \sum_{i=1}^{\ell-1} \sum_{p=0}^{n-2} \binom{n-1}{p+1} C_4^{p+1} \binom{i-1}{p} \\ &\leqq (X-b) + (X-b) \sum_{p=0}^{n-2} \binom{n-1}{p+1} C_4^{p+1} \binom{\ell}{p+1} \\ &= (X-b) \binom{n-1}{0} C_4^0 \binom{\ell-1}{0} + (X-b) \sum_{p=1}^{n-1} \binom{n-1}{p} C_4^p \binom{\ell}{p} \\ &= (X-b) \sum_{p=0}^{n-1} \binom{n-1}{p} C_4^p \binom{\ell}{p} . \end{aligned}$$

Therefore (II.2) holds for $i=n$. Q.E.D.

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On the Finite Element Approximation of Parabolic Equations —— Consistency, Boundedness, and Convergence

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Introduction

An operator theoretical proof of the convergence of approximate solutions of the parabolic equation obtained by finite element method will be presented in this article. For this purpose a variant of Trotter - Kato's approximation theory of continuous semi-groups will be summarized in §1. In §2 an abstract evolution equation, an abstract form of 2nd order parabolic differential equations, will be considered. In §3 an approximation method corresponding to the lumping method will be discussed. Finally a simple model is illustrated in §4.

The method, which will be mentioned here, are also applicable to the evolution equation of hyperbolic type. This problem has been treated by the author [7], where detailed proofs of the abstract theorems also have been written.

For the finite element approximation of time dependent problems, there have been many proofs of the convergence of approximate solution (for example Fujii [1], Strang - Fix [5], Kikuchi [3]). There would be, however, some reasons to report this note. One of the significant reasons is that the notion of consistency in the finite element approximation will be clarified operator-theoretically. In fact we will observe that the equivalence theorem of Lax holds for the present problem under suitable interpretations. Henceforth the L^2-convergence is an immediate consequence of the convergence for the stationary problem and the stability. Perturbation problems are also treated in our setting, which is illustrated in §2.

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§1. An approximation theory for semi-groups of linear operators.

Let X be a Banach space. The totality of bounded linear operators is denoted by L(X). In this article a C_0-semi-group $T(t) \in L(X)$ $(t \geq 0)$ is simply called a continuous semi-group. (As for details of the semi-group theory, see Yosida [8], Kato [2], and Krein [4].) An L(X)-valued step function $T(t)$ $(t \geq 0)$ is called a discrete semi-group with time unit τ $(\tau > 0)$ if there exists an operator $T(\tau) \in L(X)$ satisfying

$$T(t) = T(\tau)^{[t/\tau]} \quad \text{for } t \geq 0$$

where [ ] denotes the Gaussian bracket. The generator of a discrete semi-group $T(t)$ is defined by

$$A = \tau^{-1}(T(\tau) - 1).$$

A sequence of Banach spaces $\{ X_n : n=1, 2, \cdots \}$ is said to K-converge (or converge in the sense of Kato) to a Banach space X ($X_n \xrightarrow{K} X$, in short) if there exist approximating operators $P_n \in L(X,X_n)$ satisfying the following conditions (K. 1) and (K. 2):

(K. 1) $\lim_{n\to\infty} \|P_n x\| = \|x\|$ for any $x \in X$.

(K. 2) Any $x_n \in X_n$ can be expressed as $x_n = P_n x^{(n)}$ with some $x^{(n)} \in X$ satisfying $\|x^{(n)}\| \leqq N\|x_n\|$, where N is independent of n.

Now we fix a sequence of Banach spaces $\{X_n\}$ which K-converges to a Banach space X. A sequence $\{ x_n \in X_n \}$ is said to K-converge to a point $x \in X$ ($x_n \xrightarrow{K} x$, in short) if $\lim_{n\to\infty} \| x_n - P_n x \| = 0$, and sequences $\{ x_{\lambda,n} \in X_n \}_{\lambda \in \Lambda}$ are said to K-converge to points $x_\lambda \in X$ uniformly in $\lambda \in \Lambda$ if $\lim_{n\to\infty}\|x_{\lambda,n} - P_n x\| = 0$ hold uniformly in $\lambda \in \Lambda$. A sequence $\{ A_n \in L(X_n) \}$ is said to K-converge to an operator $A \in L(X)$ ($A_n \xrightarrow{K} A$, in short) if $A_n P_n x \xrightarrow{K} Ax$ for any $x \in X$, and sequences $\{ A_{\lambda,n} \in L(X_n) \}_{\lambda \in \Lambda}$ are said to K-converge to operators $A_\lambda \in L(X)$, if $A_{\lambda,n} P_n x \xrightarrow{K} A_\lambda x$ uniformly in $\lambda \in \Lambda$ for any $x \in X$.

Let us fix a continuous semi-group $T(t) \in L(X)$. And let A be its generator. Suppose that there is either a sequence of continuous semi-groups $T_n(t) \in L(X_n)$ or a sequence of discrete semi-groups $T_n(t) \in L(X_n)$ with time unit τ_n. Let A_n be the generator of semi-group $T_n(t)$. When the discrete semi-groups are considered, it is always assumed that $\lim_{n\to\infty} \tau_n = 0$.

Consider the following three conditions:

(A) (Consistency). For some complex number λ , there exist $(\lambda - A_n)^{-1} \in L(X_n)$ $(n=1, 2, \cdots)$ and $(\lambda - A)^{-1} \in L(X)$ satisfying

$$(\lambda - A_n)^{-1} \xrightarrow{K} (\lambda - A)^{-1}.$$

(B) (Boundedness).

$$\sup_{n,\ 0 \leqq t \leqq 1} \| T_n(t) \| < \infty .$$

(C) (Convergence). For any $T < \infty$

$$T_n(t) \xrightarrow{K} T(t) \text{ uniformly in } t \in [0, T].$$

The following result is fundamental in this study.

Theorem 1. (A-B-C Theorem). The conditions (A) and (B) hold if and only if the condition (C) holds.

In case $X_n \equiv X$ and $P_n \equiv I$, Theorem 1 is a corollary of Trotter - Kato's theory of approximation of semi-groups (Cf. Trotter [6], Chapter IX of Kato[1]). The notion of K-convergence is suggested in [1]. One can easily obtain the proof of Theorem 1 if he modifies Kato's treatment in [1] appropriately.

§2. A convergence proof of the approximate solutions of the evolution equation of parabolic type.

Let X and Y be Hilbert spaces. Let T be a closed operator, whose domain D(T) is dense in X, and whose range R(T) is contained in Y. It is assumed that it holds for some $\delta > 0$

$$\| Tu \| \geqq \delta \| u \| \quad \text{for any } u \in D(T).$$

The set $D(T)$ can be regarded as a Hilbert space X^1 with the inner product $(u, v)_1 = (Tu, Tv)$. As for the notational convention the space X will be denoted by X^0 sometimes. The operator $A = T^*T$ becomes a positive definite selfadjoint operator in X, of which square root $A^{1/2}$ satisfies

$$A^{1/2} \geqq \delta \quad , D(A^{1/2}) = D(T) \text{ and } \| A^{1/2}u \| = \| Tu \| .$$

Let B be a closed operator in X, whose domain $D(B)$ contains $D(T)$. Hence there is a constant β satisfying that

$$\| Bu \| \leqq \beta \| A^{1/2}u \| \quad \text{for} \quad u \in X^1. \tag{2. 1}$$

Consider the following evolution equation:

$$\text{(E)} \quad \begin{cases} \frac{d}{dt}u + Au + Bu = 0, & t > 0, \\ u(0) = u_0 \in X. \end{cases}$$

Let us define a bilinear form $c(u, v)$ with the domain $D(c) = D(T)$ as follows:

$$c(u, v) = -(Tu, Tv) - (Bu, v) \quad \text{for } u, v \in D(c).$$

This form is a closed sectorial form (Cf. Chapt. VI of Kato [1]). Therefore the operator $C = -A - B$ with domain $D(C) = D(A)$ generates an analytic semi-group $T(t)$ which satisfies for some real ω

$$\| T(t) \| \leqq e^{t\omega} \quad \text{for } t \geqq 0. \tag{2. 2}$$

The function $u(t) = T(t)u_0$ is said to be the generalized solution of (E).

To treat approximate problems we impose the following assumption.

Assumption. For any $h > 0$ there is a closed subspace X_h of X contained in $D(T)$. Let P_h^i be the orthogonal projection onto X_h in the space X^i $(i = 0, 1)$. Then it holds

$$\lim_{h\to 0} \| P_h^i u - u \|_{X^i} = 0 \quad \text{for } u \in X^i \ (i=0,\ 1).$$

Considering the set X_h as a Hilbert space having the inner product induced from the space X, we have that $X_h \xrightarrow{K} X$ with approximating operators P_h^0. Let us define the bounded selfadjoint operator $A_h \in L(X_h)$ by the formula:

$$(A_h u_h,\ v_h)_X = (Tu_h,\ Tv_h)_Y \quad \text{for any } u_h,\ v_h \in X_h.$$

The spectrum of the operator A_h is contained in the closed interval $[\ \delta^2,\ \alpha_h]$ where $\alpha_h = \| A_h \|$. The operator B is approximated by the operators $B_h = P_h^0 B$. Namely the operator B_h is defined by the formula:

$$(B_h u_h,\ v_h)_X = (Bu_h,\ v_h)_X \quad \text{for any } u_h,\ v_h \in X_h.$$

The estimate (2. 1) implies immediately

$$(2.\ 3) \qquad \| B_h u_h \| \leq \beta \| A_h^{1/2} u_h \| \quad \text{for } u_h \in X_h$$

since $\| A_h^{1/2} u_h \| = \|Tu_h \|$ for $u_h \in X_h$.

Now we consider the following approximate problem:

$$(E_h) \begin{cases} \frac{d}{dt} u_h + A_h u_h + B_h u_h = 0, \ t > 0, \\ u_h(0) = u_{h0} \in X_h. \end{cases}$$

By the same reason as the estimate (2. 2) holds, the continuous semi-group $T_h(t)$ satisfies

$$\| T_h(t) \| \leq e^{t\omega} \quad \text{for } t \geq 0.$$

It must be noted that ω can be taken independent of h because of the inequality (2. 3). Therefore the condition (B) holds for $\{ T_h(t) \}$. Let us proceed to check the condition (A). First we note that $A_h^{-1} \xrightarrow{K} A^{-1}$. In fact, let u_h and u satisfy

$$A_h u_h = P_h^0 f \text{ and } Au = f$$

for $f \in X$. Since $u_h = P_h^1 u$, we have

$$\| u_h - u \|_X \leq \delta^{-1} \|T(u_h - u)\|_Y = \delta^{-1} \| u_h - u \|_{X^1} .$$

The right hand side converges to 0 by Assumption, which implies $A_h^{-1} \xrightarrow{K} A^{-1}$. Since $\| e^{-tA_h} \| \leq 1$, we have $e^{-tA_h} \xrightarrow{K} e^{-tA}$, which in turn implies $(\lambda + A_h)^{-1} \xrightarrow{K} (\lambda + A)^{-1}$ for any $\lambda > 0$. Next we show

$$(2.\ 4) \qquad B_h(\lambda + A_h)^{-1} \xrightarrow{K} B(\lambda + A)^{-1} \quad \text{for } \lambda > 0.$$

In fact,

$$\begin{aligned}
&\| B_h(\lambda + A_h)^{-1} P_h^0 f - P_h^0 B(\lambda + A)^{-1} f \| \\
&= \| P_h^0 B \{(\lambda + A_h)^{-1} P_h^0 f - (\lambda + A)^{-1} f\} \| \\
&\leq \beta \| T \{(\lambda + A_h)^{-1} P_h^0 f - (\lambda + A)^{-1} f\} \| \\
&= \beta \| T \{A_h^{-1}(1 - \lambda(\lambda + A_h)^{-1}) P_h^0 f - A^{-1}(1 - \lambda(\lambda + A)^{-1}) f\} \| \\
&\leq \beta \| T \{A_h^{-1} P_h^0 f - A^{-1} f\} \| \\
&+ \lambda\beta \| T \{A_h^{-1}[(\lambda + A_h)^{-1} P_h^0 f - P_h^0(\lambda + A)^{-1} f]\} \| \\
&+ \lambda\beta \| T \{A_h^{-1} P_h^0(\lambda + A)^{-1} f - A^{-1}(\lambda + A)^{-1} f\} \| .
\end{aligned}$$

In the first term, set $u = A^{-1}f$. Then $u_h = A_h^{-1} P_h^0 f = P_h^1 u$. Therefore this term converges to zero by Assumption. The third term converges to zero by the same reason (set $u = (\lambda + A)^{-1} f$).

The second term

$$\begin{aligned}
&= \lambda\beta \| A_h^{-1/2} \{(\lambda + A_h)^{-1} P_h^0 f - P_h^0(\lambda + A)^{-1} f\} \| \\
&\leq \frac{\lambda\beta}{\delta} \| (\lambda + A_h)^{-1} P_h^0 f - P_h^0(\lambda + A)^{-1} f \|
\end{aligned}$$

which tends to zero since $(\lambda + A_h)^{-1} \xrightarrow{K} (\lambda + A)^{-1}$.

Finally we note that

$$(2.\ 5) \qquad \lim_{\lambda \to \infty} \| B_h(\lambda + A_h)^{-1} \| = 0 \text{ uniformly in } h.$$

In fact,

$$\begin{aligned}
\| B_h(\lambda + A_h)^{-1} \| &\leq \beta \| A_h^{1/2}(\lambda + A_h)^{-1/2} \| \\
&\leq \beta \sup_{\mu \geq \delta} \frac{\mu^{1/2}}{\lambda + \mu} \leq \frac{\beta}{2\lambda^{1/2}}
\end{aligned}$$

Noticing above mentioned two facts (2. 4), (2. 5) and the expansion formula of resolvents:

$$(\lambda - C_h)^{-1} = (\lambda + A_h + B_h)^{-1}$$
$$= \sum_{n=0}^{K} (-1)^n (\lambda + A_h)^{-1} \{B_h(\lambda + A_h)^{-1}\}^n + R_h^{K+1}(\lambda),$$

we can conclude that for any sufficiently large λ , it holds

$$(\lambda - C_h)^{-1} \xrightarrow{K} (\lambda - C)^{-1}.$$

Therefore A-B-C Theorem implies the K-convergence of $T_h(t)$ to $T(t)$.

Now we proceed to the discrete approximation defined by the following explicit scheme:

$$(E_h^{\tau_h}) \begin{cases} u_h(t + \tau_h) = (1 - \tau_h A_h - \tau_h B_h) u_h(t) \\ \qquad : k\tau_h \leqq t < (k+1)\tau_h,\ k = 0, 1, 2, \cdots, \\ u_h(t) = u_{0h} \in X_h \quad : 0 \leqq t < \tau_h. \end{cases}$$

<u>Theorem 2.</u> Choose $\tau_h \longrightarrow 0$ such that

(2. 6) $$\sup_h \tau_h \alpha_h = \gamma < 2.$$

If u_{0h} converges to u_0, the solution $u_h(t)$ of $(E_h^{\tau_h})$ converges to the generalized solution $u(t)$ of (E) in X uniformly in $t \in [0, T]$ for any finite T. If $B \equiv 0$, $\gamma = 2$ is admissible.

<u>Proof.</u> It suffices to check the conditions (A) and (B) to the discrete semi-groups $T_h(t)$ with the time unit τ_h generated by the operators C_h. The condition (A) is already asserted as above. Now we show under the condition (2. 6)

(2. 7) $$\| u_h(t) \| \leqq e^{t\bar{\omega}} \| u_h(0) \| \quad \text{for } t \geqq 0$$

with a suitable real constant $\bar{\omega}$ independent of h. For a while we drop the suffix h.

$$\| u(t + \tau) \|^2$$
$$= \| (1 - \tau A) u(t) \|^2 + \tau^2 \| Bu(t) \|^2$$
$$- 2\tau \,\mathrm{Re}(Bu(t), u(t)) + 2\tau^2 \mathrm{Re}(Bu(t), Au(t))$$

$$\leq \| (1-\tau A)u(t)\|^2 + \tau^2 \beta^2 \| A^{1/2}u(t)\|^2$$
$$+ 2\tau\beta \| A^{1/2}u(t) \| \| u(t) \| + 2\tau\beta\gamma \| A^{1/2}u(t)\| \| u(t)\|$$
$$\leq \| (1-\tau A)u(t)\|^2 + \tau\beta^2\gamma \| u(t)\|^2$$
$$+ \tau\beta (1+\gamma)(\varepsilon \| A^{1/2}u(t)\|^2 + \varepsilon^{-1} \| u(t)\|^2),$$

where ε is an arbitrary positive number. Using the spectral representation $A = \int_{\delta^2}^{\alpha} \lambda \, dE(\lambda)$, we have

(2. 8) $\quad \| u(t + \tau)\|^2$

$$\leq \int_{\delta^2}^{\alpha} \{(1-\tau\lambda)^2 + \varepsilon\beta (1+\gamma)\cdot\tau\lambda \} \, d(E(\lambda)u(t),u(t))$$
$$+ \tau (\beta^2\gamma + \frac{\beta(1+\gamma)}{\varepsilon}) \| u(t)\|^2 .$$

If $\beta \neq 0$, there is an $\varepsilon_0 > 0$ such that for any $\varepsilon \leq \varepsilon_0$, the relation ; $0 < \tau\lambda \leq \gamma$ $(0 < \gamma < 2)$ implies the relation ; $(1-\tau\lambda)^2 + \varepsilon\beta (1+\gamma) \tau\lambda \leq 1$. Fix such an ε. Then the inequality (2. 8) implies

$$\| u(t+\tau)\|^2 \leq (1+2\tau\bar{\omega}) \| u(t)\|^2$$

for some $\bar{\omega}$, which implies (2. 7). If $\beta = 0$, then the relation $0 \leq \tau\lambda \leq 2$ implies the relation $(1-\tau\lambda)^2 \leq 1$. In this case (2. 7) holds with $\bar{\omega} = 0$.

§3. Approximation of lumped mass type.

For simplicity, we restrict our attention to the equation (E) in §2 with $B = 0$. Assume there are closed subspaces $\bar{X}_h$ $(h > 0)$ satisfying the following conditions.

(L. 1) There are operators $J_h \in L(X_h, \bar{X}_h)$ and $K_h \in L(\bar{X}_h, X_h)$ such that K_hJ_h and J_hK_h are the identities on X_h and $\bar{X}_h$, respectively.

(L. 2) There is a constant N independent of h such that

$$\| J_h\| , \|K_h \| \leq N.$$

(L. 3) For any $u \in X$

$$\lim_{h \to 0} \| J_h^* J_h P_h^0 u - u \|_0 = 0,$$

where the adjoint operator J_h^* is defined by the formula

$$(J_h u, v)_X = (u, J_h^* v)_X \quad \text{for } u \in X_h \text{ and } v \in \overline{X}_h.$$

Here we introduce another approximate equation :

$$(\hat{E}_h) \quad \begin{cases} \frac{d}{dt} J_h^* J_h u_h(t) + A_h u_h(t) = 0, \\ u_h(0) = u_h \in X_h, \end{cases}$$

or equivalently for $\overline{u}_h(t) = J_h u_h(t)$,

$$(\overline{E}_h) \quad \begin{cases} \frac{d}{dt}\overline{u}_h(t) + \overline{A}_h \overline{u}_h(t) = 0, \\ \overline{u}_h(0) = \overline{u}_h \in \overline{X}_h, \end{cases}$$

where the operator $\overline{A}_h = K_h^* A_h K_h$ is a positive definite bounded self-adjoint operator in $\overline{X}_h$.

In these situations, we have $\overline{X}_h \xrightarrow{K} X$ with approximating operators $\overline{P}_h = J_h P_h^0$.

Fix $f \in X$ arbitrarily. Let u, u_h and $\overline{v}_h$ be the solutions of $Au = f$, $A_h u_h = P_h f$ and $\overline{A}_h \overline{v}_h = \overline{P}_h f$. We have

$$\| \overline{v}_h - \overline{P}_h u \| \leq \| \overline{v}_h - J_h P_h^1 u \| + \| J_h P_h^1 u - J_h P_h^0 u \| .$$

The 2nd term converges to 0 as h tends to 0 by the condition (L. 2) and Assumption in §2. As for the 1st term it must be noted that $P_h^1 u = u_h$, and that $A_h v_h = J_h^* J_h P_h^0 f$ for $v_h = K_h \overline{v}_h$. Hence

$$\begin{aligned} M_h &= \| \overline{v}_h - J_h P_h^1 u \| = \| \overline{v}_h - J_h u_h \| \\ &\leq (\delta N)^{-1} \| \overline{A}_h^{1/2} (\overline{v}_h - J_h u_h) \| \\ &= (\delta N)^{-1} \| A_h^{1/2} (v_h - u_h) \| \\ &= (\delta N)^{-1} \| A_h^{-1/2} (J_h^* J_h P_h^0 f - P_h^0 f) \| \\ &\leq \delta^{-2} N^{-1} (\| J_h^* J_h P_h^0 f - f \| + \| f - P_h^0 f \|) \end{aligned}$$

$\longrightarrow 0$ (by (L. 3) and Assumption).

Therefore $\bar{A}_h^{-1} \xrightarrow{K} A^{-1}$. Since $\| e^{-t\bar{A}_h} \| \leqq 1$, A-B-C Theorem asserts that $e^{-t\bar{A}_h} \xrightarrow{K} e^{-tA}$ locally uniformly in $t \geqq 0$.

As for the discrete approximation, the corresponding result to Theorem 2 is obtained if we replace α_h with $\| \bar{A}_h \|$. The perturbed equation can be also treated. Finally we remark that the condition (L. 3) can be replaced with the following condition (Cf. Kikuchi [3]).

(L. 3)' There is a function $\varepsilon(h)$ tending to 0 as h tends to 0 satisfying

$$\| J_h u_h - u_h \|_0 \leqq \varepsilon(h) \| u_h \|_1 \quad \text{for any } u_h \in X_h.$$

In fact,

$$\begin{aligned} M_h^2 &= (A_h(v_h-u_h),\ v_h-u_h) \\ &= (P_h^0 f - J_h P_h^0 f,\ v_h-u_h) \\ &\qquad + (J_h P^0{}_h f,\ v_h - J_h v_h) \\ &\qquad - (J_h P^0{}_h f,\ u_h - J_h u_h) \\ &= I + J + K. \end{aligned}$$

It is noted that $\| v_h \|_0$, $\| u_h \|_0$, $\| v_h \|_1$, $\| u^1{}_h \|_1$ are bounded by a finite constant C independent of h. Therefore we have

$$| I | \leqq 2C \| P_h^0 f - J_h P_h^0 f \|,$$

which converges to 0 by the condition (L. 3)' and Banach - Steinhauss Theorem. Also we have by the condition (L. 3)'

$$| J |,\ | K | \leqq N \| f \| \varepsilon(h) C \longrightarrow 0.$$

§4. Space 1 dimensional parabolic equation — an illustration.

Let us consider the following initial boundary value problem.

$$(F)\quad\begin{cases}\dfrac{\partial}{\partial t}u(t, x) = \left(\dfrac{\partial}{\partial x}a(x)\dfrac{\partial}{\partial x} + b(x)\dfrac{\partial}{\partial x} + c(x)\right)u(t, x) \\ \qquad : \ t > 0,\ 0 < x < 1, \\ \left(\dfrac{\partial}{\partial x} - \sigma_0\right)u(t, 0) = \left(\dfrac{\partial}{\partial x} + \sigma_1\right)u(t, 1) = 0 \quad : t > 0, \\ u(0, x) = u(x),\end{cases}$$

where $a(x)$ is a positive function belonging to $C^1([0, 1])$, $b(x)$ and $c(x)$ are bounded measurable functions, σ_0 and σ_1 are positive constants. Let us denote the space $L_2((0, 1))$ by X, and the space $L^2((0, 1), a(x)dx)$ (weighted L^2-space) by Y_0, and the space C^2 with the inner product $\left(\begin{pmatrix}\xi_0\\ \xi_1\end{pmatrix}, \begin{pmatrix}\eta_0\\ \eta_1\end{pmatrix}\right) = a(0)\,\sigma_0\,\xi_0\,\overline{\eta}_0 + a(1)\,\sigma_1\,\xi_1\,\overline{\eta}_1$ by Y_1. Let Y be the product Hilbert space $Y_0 \times Y_1 \times X$. Consider the following closed operator T from X into Y:

$$Tu = \left(\frac{d}{dx}u(x),\ \begin{pmatrix}u(0)\\ u(1)\end{pmatrix},\ u(x)\right) \quad \text{for } u \in D(T) = H^1(0, 1).$$

Then we have

$$T^*w = -\frac{d}{dx}(a(x)v(x)) + u(x) \quad \text{for } w \in D(T^*)$$

where

$$D(T^*) = \left\{ w = \left(v, \begin{pmatrix}-v(0)/\sigma_0\\ v(1)/\sigma_1\end{pmatrix}, u\right) : v \in H^1(0, 1),\ u \in L^2(0, 1) \right\}.$$

Then the operator $A = T^*T$ can be regarded as the realization in $L^2(0, 1)$ of the differential operator $-\frac{d}{dx}(a(x)\frac{d}{dx}) + 1$ with boundary conditions $(\frac{d}{dx} - \sigma_0)u(0) = (\frac{d}{dx} + \sigma_1)u(1) = 0$. Define

$$Bu = -b(x)\frac{d}{dx}u(x) - (c(x)+1)u(x) \quad \text{for } u \in D(B) = H^1(0, 1).$$

Then the equation (F) is reduced to the equation (E).

Now let us consider the simplest case — approximation by piece-wise linear functions with equal mesh size. Set

$$\lambda(x) = \begin{cases} |1 - x| & : \ |x| \leq 1, \\ 0 & : \ |x| > 1, \end{cases}$$

$$\chi(x) = \begin{cases} 1 & : \ |x| \leq 1/2, \\ 0 & : \ |x| > 1/2. \end{cases}$$

For $h = 1/n$ (n is an integer), put

$$\lambda_j^h(x) = \lambda\left(\frac{x-jh}{h}\right), \quad \bar{\lambda}_j^h(x) = \lambda\left(\frac{x-jh}{h}\right),$$

and

$$X_h = \left\{ \sum_{j=0}^{n} \phi_j^h \lambda_j^h \right\}, \quad \bar{X}_h = \left\{ \sum_{j=0}^{n} \phi_j^h \bar{\lambda}_j^h \right\}.$$

Then the spaces X_h satisfy Assumption in §2. Let us define

$$J_h\left(\sum_{j=0}^{n} \phi_j^h \lambda_j^h\right) = \sum_{j=0}^{n} \phi_j^h \bar{\lambda}_j^h,$$

$$K_h\left(\sum_{j=0}^{n} \phi_j^h \bar{\lambda}_j^h\right) = \sum_{j=0}^{n} \phi_j^h \lambda_j^h.$$

Then the pairs $(X_h, \bar{X}_h, J_h, K_h)$ satisfy the conditions (L. 1) to (L. 3). If $a(x) \equiv 1$, we have

$$\| A_h{}^{1/2} \| \leq 2\sqrt{3}/h + \sqrt{6\max(\sigma_0, \sigma_1)}/\sqrt{h} + 1.$$

$$\| \bar{A}_h{}^{1/2} \| \leq 2/h + \sqrt{2\max(\sigma_0, \sigma_1)}/\sqrt{h} + 1.$$

Therefore if $\tau_h/h^2 < 1/2$, the solution of $(\bar{E}_h^{\tau_h})$ converge to the solution of (E). (The equation $(\bar{E}_h^{\tau_h})$ is obtained from the equation $(E_h^{\tau_h})$ in §2 after replacement: $u_h \longrightarrow \bar{u}_h$, $A_h \longrightarrow \bar{A}_h$ and $B_h \longrightarrow \bar{B}_h = \bar{P}_h B$.) Incidentally it is remarked that the present difference equation $(\bar{E}_h^{\tau_h})$ is just the explicit difference approximation of the equation (F).

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Numerical Solution of the Stefan Problem by the Finite Element Method

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1. Introduction

The equations describing a typical Stefan-type free boundary problem for heat equations in one dimension will be stated as follows. The main equation for temperature $u(t,x)$ is

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} , \quad 0 < x < s(t) , \quad 0 \leq t \leq T , \tag{1.1}$$

where $s(t)$ is the position of the boundary which moves because of melting or freezing of the material, for example. The initial condition is given

$$s(0) = a \tag{1.2}$$

$$u(0,x) = g(x) \geq 0 , \quad 0 \leq x \leq a . \tag{1.3}$$

At $x = s(t)$ the boundary condition is given

$$u(t,s(t)) = 0 , \tag{1.4}$$

and the boundary moves according to the following equation which is called the Stefan condition.

(1.5) $$\frac{ds}{dt} = - \kappa \frac{\partial u}{\partial x} (t,s(t)) \ .$$

At $x = 0$ we assume

(1.6) $$u(t,0) = f(t) \geq 0 \ .$$

Almost all of the works on numerical solutions of one dimensional Stefan problems have been carried out by using some difference scheme in a rectangular lattice in x-t space in which the mesh size of space variable x and that of the time variable t are kept fixed throughout the computation [1,2,3,4]. Landau [5] proposed another difference scheme by normalizing the domain $0 \leq x \leq s(t)$ by introducing a new variable $\xi = x/s(t)$ and partitioning the normalized domain into equal subintervals.

In the present paper we propose a new method based on the finite element method (FEM) with time dependent basis functions, which will turn out to be applicable to a large class of problems having moving or free boundaries.

2. Application of the Finite Element Method to the One Dimensional Problem

In FEM for initial value problems the partition of the domain is usually fixed throughout the computation. In contrast to that the domain is partitioned anew at each time t in our method in such a way that the position $s(t)$ of the boundary always coincides with the end point of the partition.

Consider the domain $0 \leq x \leq s(t)$ at time t. We divide $0 \leq x \leq s(t)$ into n equal subintervals and denote each node as

$$(2.1) \quad x_j = jh , \quad j = 0,1,\dots,n; \quad h = h(t) = \frac{s(t)}{n} .$$

Then we construct piecewise linear basis functions $\{\phi_j(t,x)\}$ according to the standard prescription, where the suffix j corresponds to x_j (Fig. 1):

$$(2.2) \qquad \phi_j(t,x) = \begin{cases} \frac{1}{h} x-(j-1) \; ; & (j-1)h \leq x < jh \\ -\frac{1}{h} x+(j+1) \; ; & jh \leq x < (j+1)h \; ; \end{cases}$$

$$j = 1,2,\dots,n-1.$$

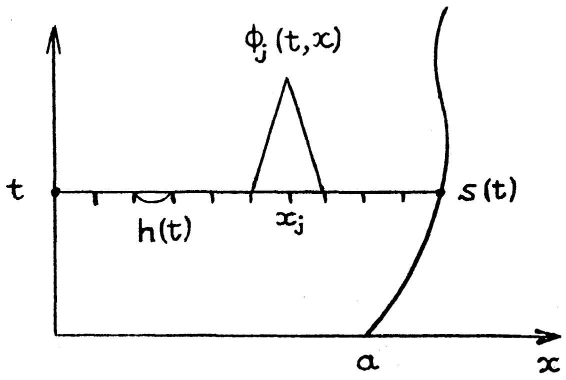

Fig. 1

$\phi_j(t,x)$ vanishes outside $(j-1)h \leq x < (j+1)h$. Note that $\phi_j(t,x)$ depends not only on x but also on t because of $h = h(t)$.

Now we apply the Galerkin method based on the basis functions $\{\phi_j(t,x)\}$. We expand the approximation $v(t,x)$ of the solution $u(t,x)$ in terms of $\phi_j(t,x)$:

$$v(t,x) = \sum_{j=1}^{n-1} b_j(t)\,\phi_j(t,x) + f(t)\,\phi_0(t,x)\ . \tag{2.3}$$

Substituting (2.3) into $u(t,x)$ of (1.1), multiplying $\phi_k(t,x)$ and integrating over $(0,s(t))$, we have

$$M\frac{d\vec{b}}{dt} + N\vec{b} = -K\vec{b} + \vec{f}\ , \tag{2.4}$$

where $M = M(t)$ and $K = K(t)$ are mass matrix and stiffness matrix, respectively:

$$M_{jk} = \int_0^s \phi_j\,\phi_k\,dx \tag{2.5}$$

$$K_{jk} = \int_0^s \frac{\partial\phi_j}{\partial x}\frac{\partial\phi_k}{\partial x}\,dx\ . \tag{2.6}$$

In addition to those matrices, time dependence of $\phi_j(t,x)$ gives rise to another matrix $N = N(t)$ in (2.4):

$$N_{jk} = \int_0^s \phi_j \frac{\partial \phi_k}{\partial t}\, dx \,, \tag{2.7}$$

where

$$\frac{\partial \phi_k}{\partial t} = \begin{cases} -\dfrac{1}{h^2}\dfrac{dh}{dt}\, x \,; & (k-1)h \le x < kh \\[2ex] \dfrac{1}{h^2}\dfrac{dh}{dt}\, x \,; & kh \le x < (k+1)h \,. \end{cases} \tag{2.8}$$

$\partial\phi_k/\partial t$ vanishes outside $(k-1)h \le x < (k+1)h$, so that the sparseness of N is the same as that of M and K. The explicit expressions of the non-zero elements of M, K and N are obtained as follows:

$$M_{jj} = \frac{2}{3}h \;(j \ne 1), \quad M_{11} = \frac{1}{3}h \,, \quad M_{jj\pm1} = \frac{1}{6}h \tag{2.9}$$

$$K_{jj} = \frac{2}{h} \;(j \ne 1) \,, \quad K_{11} = \frac{1}{h} \,, \quad K_{jj\pm1} = -\frac{1}{h} \tag{2.10}$$

$$N_{jj} = \frac{1}{3}\frac{dh}{dt} \;(j \ne 1) \,, \; N_{11} = \frac{1}{6}\frac{dh}{dt} \,, \tag{2.11}$$

$$N_{jj-1} = \frac{1}{6}(3j-1)\frac{dh}{dt} \,, \; N_{jj+1} = -\frac{1}{6}(3j+1)\frac{dh}{dt} \,.$$

All the elements other than those mentioned above vanish. Note that N is not symmetric. The j-th element of the vector $\vec{f}$ is given by

$$\vec{f}_j = f(t)\int_0^s \frac{\partial \phi_0}{\partial t}\phi_j\, dx + f'(t)\int_0^s \phi_0\,\phi_j\, dx \,, \tag{2.12}$$

where we assume that $f(t)$ is differentiable. $\vec{b}$ is the unknown vector, the j-th element of which is $b_j(t)$.

We obtain various kinds of schemes suitable for numerical computation if we approximate the time derivative of $\vec{b}$ by the time difference $\{\vec{b}(t+\Delta t) - \vec{b}(t)\}/\Delta t$, i.e.

$$(2.13)\quad \{M(t+\tfrac{1}{2}\Delta t) + \theta\,\Delta t\,N(t+\tfrac{1}{2}\Delta t) + \theta\,\Delta t\,K(t+\tfrac{1}{2}\Delta t)\}\vec{b}(t+\Delta t)$$

$$= \{M(t+\tfrac{1}{2}\Delta t) - (1-\theta)\Delta t N(t+\tfrac{1}{2}\Delta t) - (1-\theta)\Delta t K(t+\tfrac{1}{2}\Delta t)\}\vec{b}(t)$$

$$- \vec{f}(t+\tfrac{1}{2}\Delta t)\ , \qquad 0 \le \theta \le 1,$$

where $\theta = 0$ and $\theta = 1$ correspond to the forward and backward difference approximation, respectively.

We approximate the Stefan condition by

$$(2.14)\qquad \frac{ds}{dt}(t) \doteqdot \frac{\kappa}{h}\, b_{n-1}(t)$$

Higher order approximations for $\frac{\partial u}{\partial x}(t,s(t))$ will also be obtainable if we pick up the values of v at other sampling points in the neighborhood of the boundary.

Now we are ready to write down the whole procedure. Initially put

$$(2.15)\qquad b_j(0) = g(x_j)\ ,$$

and compute $s(\Delta t/2)$ by

$$(2.16) \qquad s(\Delta t/2) = a + \frac{\kappa n}{2a} \Delta t \; g(x_{n-1}) \; .$$

Suppose $\vec{b}(t)$ and $s(t+\Delta t/2)$ are obtained. Compute all the elements necessary for (2.13) using $h(t+\Delta t/2) = s(t+\Delta t/2)/n$, and solve (2.13) for $\vec{b}(t+\Delta t)$. Then compute $s(t+3\Delta t/2)$ by

$$(2.17) \qquad s(t + \tfrac{3}{2}\Delta t) = s(t + \tfrac{1}{2}\Delta t) + \frac{\kappa}{h} \Delta t \; b_{n-1}(t) \; .$$

We tried to apply our method to an example problem which several authors have dealt with [3,4]:

$$(2.18) \qquad f(t) = \cos \frac{\pi}{4} t \, , \quad 0 \le t \le 2$$

$$(2.19) \qquad g(x) = 1 - x \; .$$

The number of partitions and the time mesh size are

$$(2.20) \qquad n = 2^4 = 16$$

$$(2.21) \qquad \Delta t = \frac{1}{2^{11}} = \frac{1}{2048} \; .$$

We applied the forward difference scheme ($\theta = 0$) and obtained a result which agrees well with those shown in [3,4].

The contribution of $N(t)$ to the solution $\vec{b}$ with a very small meshsize of Δt is fairly small compared with that of $K(t)$, so that, roughly speaking, the stability of the present scheme is considered to be guaranteed if the scheme without N is a stable one. In fact, we obtained a reasonable result with $\lambda = n^2 \Delta t = 1/8 < 1/6$. On the other hand, we observed some instability in the computation with $\lambda = 1/4 > 1/6$ which violates the stability condition of the scheme with $\theta = 0$ for the usual heat equation in a fixed domain [6].

3. Discussion

The present idea is applicable also to the finite difference method. In fact, in the one-dimensional case, an FEM for our problem will correspond to a certain kind of finite difference method similar to that proposed by Landau [5]. But the merit of our idea will be more evident when applied together with FEM, because FEM is considered to be easier to apply to problems in two or three dimensions. In a succeeding paper we will show some results of applications of our method to two-dimensional Stefan problems.

In the present method it is essential that each node be determined uniquely as a function of the position s of the boundary, so that in the case of the two- or three-dimensional problem, it would be necessary for the boundary to be assumed to be kept star-shaped.

Although we took a partition into equal size subintervals in §2, it is not at all necessary to do so. What we need to notice is that each node is uniquely determined as a function of s. We can choose some suitable partition according to the nature of the problem.

The fact that every matrix element must be computed anew at each time step might be seen to be a drawback in the present method. But it is not the case because each matrix element depends on time as a simple function of t and so the computation of the matrix elements at each time step is not serious. In fact, in the example show in §2, $M(t)$, $K(t)$ and $N(t)$ are constant matrices multiplied by scalar functions of t.

Finally the author expresses his thanks to Miss Tomoko Takashashi for her help in the numerical calculations.

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On the Numerical Solvability of Two Point Boundary Value Problems in a Finite Chebyshev Series for Piecewise Smooth Differential Systems

Kumiko ITOH

In his paper [2], Urabe has proved the existence of a good approximation in a finite Chebyshev series to the exact solution to the boundary value problem for differential systems of the form :

$$(*) \qquad \frac{dx}{dt} = X(x,t),$$

where x and $X(x,t)$ are vectors of the same dimension. In [2], however, it is assumed that the function $X(x,t)$ is twice continuously differentiable with respect to x and t in the domain of $X(x,t)$.

In the present paper, the author considers two point boundary value problems for the differential system (*) under the assumption that $X(x,t)$ is piecewise twice continuously differentiable with respect to x and t, that is, it is continuous with respect to x and t and in addition it is twice continuously differentiable with respect to x and t in the interior of each region of the subdivision of the domain.

The value of t at which the trajectory of a desired solution crosses a boundary curve of a subdomain will be called the switching time. In the present paper, similarly to Mohler and Moon [1], all possible switching times will be treated as unknown parameters and a desired solution will be sought on each subinterval divided by switching times. The problem is thus reduced to a system of boundary value problems with nonlinear boundary conditions.

If a desired solution is approximated by a finite

Chebyshev series on each subinterval divided by switching times, then the numerical solution of the problem under consideration is reduced to that of a system of nonlinear algebraic equations just as the problem treated by Urabe in [2]. In [2], Urabe has proved that such a system of nonlinear algebraic equation has indeed a solution corresponding to an exact solution provided the exact solution in question is isolated. In the present paper, the author will show that the system of nonlinear algebraic equations in question has likewise a solution corresponding to an exact solution provided the exact solution in question is isolated and moreover its trajectory transverses the boundary curves of the subdomains, when it crosses.

The result of the present paper shows the numerical solvability of two-point boundary value problems in a finite Chebyshev series for piecewise smooth differential systems and hence the possibility of the numerical computation of solutions in a finite Chebyshev series.

The author wishes to acknowledge with gratitude the valuable advice and guidance she received from Professor Minoru Urabe.

§ 1. Preliminaries

Let D be a region of the tx-space (x:n-vector) intercepted by two hyperplanes t = -1 and t = 1, and D_1 be a region, which includes D in its interior. We assume that D is divided into a finite number of subregions E_λ, $\lambda \in \Lambda$, that is,

$$D = \bigcup_{\lambda\in\Lambda} \bar{E}_\lambda,$$

$$E_{\lambda_1} \cap E_{\lambda_2} = \phi, \qquad \text{for } \lambda_1 \neq \lambda_2,$$

where E_λ is a open connected set and the symbol $\bar{E}$ denotes the closure of the set E.

In addition, we assume that each $\bar{E}_\lambda$ has the following structure.

For any boundary point (t_1, x_1) of E_λ, we can find real valued twice continuously differentiable functions on D_1, $f_i(x,t)$, $i=1,\cdots,s$, such that E_λ is given by

(1.1) $\qquad f_1(x,t) < 0, \cdots, f_s(x,t) < 0,$

in a neighborhood of (t_1, x_1), and such that

(1.2) $\qquad f_i(x_1, t_1) = 0, \qquad i=1,\cdots,s.$

Let $Y_\lambda(x,t)$, $\lambda \in \Lambda$ be continuous n-vector functions on D. We assume that $Y_\lambda(x,t)$ is twice continuously differentiable with respect to x and t in D, and that

(1.3) $\quad Y_{\lambda_1}(x,t) = Y_{\lambda_2}(x,t), \quad$ for $(t,x) \in \bar{E}_{\lambda_1} \cap \bar{E}_{\lambda_2}$.

Let X(x t) be a piecewise smooth function defined by

(1.4) $\quad X(x,t) = Y_\lambda(x,t), \quad$ for $(t,x) \in \bar{E}_\lambda$.

In the present paper, we consider the system of differential equations

(1.5) $\qquad \frac{dx}{dt} = X(x,t)$

with the two point boundary condition

(1.6) $\qquad L_0 x(-1) + L_1 x(1) = \ell$.

Here L_i, $i=0,1$, are given square matrices and ℓ is a given vector.

We consider the solution x(t) such that (i) the number of switching times is finite, (ii) the curve (t, x(t)) passes through the boundary expressed in only one hypersurface.

Moreover we are concerned with the solution, which is the following form :

(1.7) $\quad (t, x(t)) \in E_j, \quad$ for $t \in (t_{j-1}, t_j)$, $\quad j=1,\cdots,r,$

$$(1.8)\qquad f^j(t_j,\ x(t_j)) = 0, \quad j=1,\cdots,r-1,$$

where

$$t_1 < t_2 < \cdots < t_{r-1}, \quad t_0 = -1, \quad t_r = 1.$$

Here t_j, $j=1,\cdots,r-1$ are switching times, E_j, $j=1,\cdots,r$ subregions, and $f^j(x,t) = 0$, $j=1,\cdots,r-1$, hypersurfaces.

In what follows, however, we can include the case that $t_{j-1} = t_j$ $(2 \leqq j \leqq r-1)$ as a limit case that $(t_{j-1}-t_j) \longrightarrow 0$. Thus we replace above inequalities by

$$(1.9)\qquad t_1 \leqq t_2 \leqq \cdots \leqq t_{r-1}.$$

Let $x_j(t)$ be a restricted function of $x(t)$ on $[t_{j-1},\ t_j]$, for $j=1,\cdots,r$. Then, from (1.4)$\sim$(1.8), we have the boundary value problem with nonlinear boundary conditions,

$$(1.10)\qquad \begin{cases} \dot{x}_j(t) = Y^j(x_j(t),\ t), & j=1,2,\cdots,r, \\ f^j(x_j(t_j),\ t_j) = 0, & j=1,\cdots,r-1 \\ L_0x_1(-1) + L_1x_r(1) = l\ , & \\ x_{j+1}(t_j) = x_j(t_j), & j=1,\cdots,r-1\ , \end{cases}$$

where Y^j is a function defined in (1.4) corresponding to the subregion E_j.

We put

$$(1.11)\qquad t_j(\tau) = \frac{t_j-t_{j-1}}{2}\,\tau + \frac{t_j+t_{j-1}}{2}\ ,$$

$$x^j(\tau) = x_j(t_j(\tau)), \qquad j=1,\cdots,r.$$

Then, from (1.10), $x^j(\tau)$, $j=1,\cdots,r$, satisfy the following boundary value problem:

$$(1.12)\qquad \dot{x}^j(\tau) = Y^j(x^j(\tau),\ t_j(\tau))\frac{t_j-t_{j-1}}{2}, \quad \text{for } \tau\in[-1,1]\ ,$$

$$j=1,2,\cdots,r,$$

$$(1.13)\quad \begin{cases} f^j(x^j(1),\ t_j) = 0 & j=1,2,\cdots,r-1 \\ L^0x^1(-1) + L_1x^r(1) = \ell, & \\ x^{j+1}(-1) = x^j(1), & j=1,2,\cdots,r-1 \end{cases}$$

Thus the solutions of the boundary value problem (1.5) and (1.6) can be obtained by solving the boundary value problem (1.12) and (1.13) with respect to $x^j(\tau)$, $j=1,\cdots,r$ and switching times t_j, $j=1,2,\cdots,r-1$.

Let $f(t)$ be a continuous vector function defined on $[-1,1]$ and let its Chebyshev series be

$$(1.14)\qquad f(t) \sim a_0 + \sqrt{2}\sum_{n=1}^{\infty} a_nT_n(t).$$

We use two kinds of norms $\|f\|_n$ and $\|f\|_q$, which are defined as follows :

$$(1.15)\qquad \begin{aligned} \|f\|_n &= \sup_{t\in[-1,1]} \|f(t)\| , \\ \|f\|_q &= [\tfrac{1}{\pi}\int_0^{\pi}\|f(\cos\theta)\|^2 d\theta]^{\frac{1}{2}} , \end{aligned}$$

where the symbol $\|\ \|$ denotes the Euclidean norm. Let P_mf or f_m denotes a function discarding the terms of the order higher than m in the right hand side of (1.14). Then the following properties hold [see Ref. 2].

$$(1.16)\qquad \begin{aligned} &\|f\|_q \leqq \|f\|_n, \\ &\|P_mf\|_q = \sum_{n=0}^{m} \|a_n\|^2, \\ &\|P_mf\|_n \leqq \sqrt{2m+1}\ \|P_mf\|_q, \\ &\|f-P_mf\|_q = \frac{1}{m+1}\|\dot{f}\|_q . \end{aligned}$$

Now, for (1.12) and (1.13), let

$$x_m^j(\tau) = a_0^j + \sqrt{2}\sum_{n=1}^{m} a_n^jT_n(\tau), \quad j=1,2,\cdots,r, \ (\tau\in[-1,1])$$

be desired finite Chebyshev series of $x^j(\tau)$, $j=1,2,\cdots,r$ with undetermined coefficients a_i^j, $i=0,\cdots,m$, $j=1,2,\cdots,r$. Then the determining equation, by which unknown switching times t_i, $i=1,2,\cdots,r-1$, and Chebyshev coefficients a_i^j, $i=0,1,\cdots,m$, $j=1,2,\cdots,r$ should be determined, is as follows :

$$(1.17)\quad \begin{cases} z^0(\alpha) = \sum_{i=0}^{1} L_i x_m^{j_i}((-1)^{i+1}) - \ell = 0, \\ z^j(\alpha) = x_m^{j+1}(-1) - x_m^j(1) = 0, \quad j=1,2,\cdots,r-1, \\ z_i^j(\alpha) = \frac{2}{\pi e_{i-1}} \int_0^\pi Y^j(x_m^j(\cos\theta),\ t_j(\cos\theta))\cos(i-1)\theta \\ \qquad d\theta \frac{t_j - t_{j-1}}{2} - \frac{1}{e_{i-1}} \sum_{s=0}^{m} v_{s-(i-1)} s a_s^j = 0, \\ \qquad i=1,2,\cdots,m,\ j=1,2,\cdots,r, \\ f^j(\alpha) = f^j(x_m^j(1),\ t_j) = 0, \quad j=1,2,\cdots,r-1. \end{cases}$$

where

$$\alpha = \mathrm{col}(a_0^1 \cdots a_m^1 a_0^2 \cdots a_m^2 \cdots a_0^r \cdots a_m^r t_1 \cdots t_{r-1}),$$

$$j_0 = 1,\ j_1 = r,\ e_s = \begin{cases} \sqrt{2}, & \text{for } s=0 \\ 1, & \text{otherwise}, \end{cases} \qquad v_s = \begin{cases} 0, & \text{for } s<0, \\ 1-(-1)^s, & \text{for } s \geqq 0. \end{cases}$$

In what follows, we shall write (1.17) in a vector form as follows :

$$(1.18)\qquad F^{(m)}(\alpha) = 0.$$

The purpose of this paper is to prove the following theorem.

Theorem. We assume the following three conditions :

(a) The system (1.5) with (1.6) has an isolated solution $\hat{x}(t)$, that is, the solution such that the matrix $G = L_0 + L_1\Phi(1)$ is nonsingular. Here $\Phi(t)$ is a fundamental matrix of

$$(1.19)\qquad \frac{dy}{dt} = X_x(\hat{x}(t),\ t)y$$

such that

$$\Phi(-1) = E.$$

(b) The curve $\hat{x}(t)$ satisfies the transversality condition,

$$A_j = f_x^{j}(\hat{x}(\hat{t}_j), \hat{t}_j)X(\hat{x}(\hat{t}_j), \hat{t}_j) + f_t^j(\hat{x}(\hat{t}_j), \hat{t}_j) \neq 0, \quad j=1,2,\cdots,r-1.$$

That is, at $\hat{t}_j$, the curve $(t, x(t))$ transverses the hypersurface represented by $f^j(x,t) = 0$.

(c) The curve $\hat{x}(t)$ satisfies the internality condition

(1.20) $\quad U = \{(t,x) : \|x-\hat{x}(t)\| \leqq \delta, \quad t \in [-1,1]\} \subset D.$

Then, for sufficiently large m_0, there are finite Chebyshev series $x_m^j(\tau)$, $j=1,\cdots,r$, of any order $m \geqq m_0$, and approximate switching times t_i, $i=1,\cdots,r-1$, such that

(1.21) $\quad x_m^j(\tau) \longrightarrow \hat{x}^j(\tau)$, uniformly as $m \longrightarrow \infty$, and $t_i \longrightarrow \hat{t}_i$ as $m \longrightarrow \infty$, for $j=1,\cdots,r$, $i=1,\cdots,r-1$.

Here

$$\hat{x}^j(\tau) = \hat{x}\left(\frac{\hat{t}_j-\hat{t}_{j-1}}{2}\tau + \frac{\hat{t}_j-\hat{t}_{j-1}}{2}\right), \text{ for } \tau \in [-1,1], \quad j=1,\cdots,r.$$

The proof of Theorem is based on the following proposition.

Proposition 1. Consider a real system of algebraic equations

(1.22) $$\begin{cases} Y\alpha - \ell = 0, \\ F(\alpha) = 0, \end{cases}$$

where α is a n-vector, Y is a constant $k \times n$-matrix and $F(\alpha)$ is a twice continuously differentiable $(n-k)$-vector function defined in some region Ω of α-space. Assume that (1.22) has an approximate solution $\alpha = \bar{\alpha}$ for which the determinant of $J(\alpha) = \begin{pmatrix} Y \\ F_\alpha(\bar{\alpha}) \end{pmatrix}$ does not vanish. Further assume that there are a positive constant δ and a non-negative $\kappa < 1$ such that

(i) $\Omega_\delta = \{\alpha : \|\alpha - \bar{\alpha}\| \leqq \delta\} \subset \Omega$,

(ii) $\|F_\alpha(\alpha) - F_\alpha(\bar{\alpha})\| \leqq \frac{\kappa}{M_1}$, for any $\alpha \in \Omega_\delta$,

(iii) $\dfrac{M_1 r + M_2 \varepsilon}{1-\kappa} \leqq \delta$,

where r, ε, M_1 and M_2 are numbers such that

$$\|F(\bar{\alpha})\| = r,\ \|Y\bar{\alpha} - \ell\| = \varepsilon,\ \left\|\begin{pmatrix} Y \\ F_\alpha(\bar{\alpha}) \end{pmatrix}^{-1} \begin{pmatrix} 0 & 0 \\ 0 & E \end{pmatrix}\right\| \leqq M_1 \text{ and}$$
$$\left\|\begin{pmatrix} Y \\ F_\alpha(\bar{\alpha}) \end{pmatrix}^{-1} \begin{pmatrix} E & 0 \\ 0 & 0 \end{pmatrix}\right\| \leqq M_2, \tag{1.23}$$

where $\begin{pmatrix} 0 & 0 \\ 0 & E \end{pmatrix}$ and $\begin{pmatrix} E & 0 \\ 0 & 0 \end{pmatrix}$ are square, E is a unit $(n-k) \times (n-k)$-matrix in the former and a unit $k \times k$-matrix in the latter, and 0's are appropriate zero matrices.

Then the system of algebraic equations has the unique solution $\alpha = \hat{\alpha}$ in Ω_δ, and for $\alpha = \hat{\alpha}$, it holds that

$$\det J(\hat{\alpha}) \neq 0, \tag{1.24}$$

$$\|\bar{\alpha} - \hat{\alpha}\| \leqq (M_1 r + M_2 \varepsilon)/(1-\kappa). \tag{1.25}$$

Proposition 1 is the same with Proposition 1 in Ref. 2, except we separate the linear part and the nonlinear part in the system of algebraic equations. We omit the proof, since it is similar to the proof of Theorem in Ref. 3.

§ 2. Isolatedness, Internality condition and Truncated Polynomial of the Isolated solution.

In the present and succeeding sections, we assume the system (1.5) with (1.6) has an isolated solution $\hat{x}(t)$ with switching times $\hat{t}_i$, $i=1,2,\cdots,r-1$, of the form (1.7) with (1.9), satisfying both the internality condition for some positive δ and the transversality condition.

By the following Proposition 2, the isolatedness of $x(t)$ are given.

Proposition 2. Let $\Phi_j(\tau)$ be a fundamental matrix of

(2.1) $$\frac{dy}{d\tau} = Y_x^j(x^j(\tau),\ t_j(\tau))\frac{t_j - t_{j-1}}{2}\, y$$

satisfying

$$\Phi_j(-1) = E, \quad \text{for } j=1,\cdots,r.$$

Then $\Phi(t)$ of (1.19) is expressed as

(2.2) $$\Phi(t) = \Phi_j\left(\frac{2t-(t_j+t_{j-1})}{t_j-t_{j-1}}\right)\Phi_{j-1}(1)\cdots\Phi_1(1)$$

$$\text{on } [t_{j-1},\ t_j],\ j=1,\cdots,r,$$

and further

(2.3) $$G = L_0 + L_1\Phi_r(1)\Phi_{r-1}(1)\cdots\Phi_1(1).$$

If $t_{j-1} = t_j$, we put, in (2.2),

$$\Phi_j\left(\frac{2t-(t_j+t_{j-1})}{t_j-t_{j-1}}\right) = E \quad \text{for } t\in[t_{j-1},\ t_j].$$

Proof. Proof is derived easily by mathematical induction.

Now from both the absolutely continuity of $\hat{x}(t)$ and the compactness of the interval $[-1,1]$, it holds that $\hat{x}(t)$ is uniformly continuous. That is, for any $\varepsilon\in(0,\delta)$, there exists some positive constant δ_{02} such that

(2.4) $$\|\hat{x}(t)-\hat{x}(t+\lambda)\|_n \leqq \varepsilon, \quad \text{for any } \lambda\in[-\delta_{02},\ \delta_{02}],$$
$$(t,\ t+\lambda\in[-1,1]).$$

We take some $\varepsilon\in(0,\delta)$ and put

(2.5) $$\delta_{01} = \delta - \varepsilon.$$

Then we have the following proposition.

Proposition 3. For any functions $x^j(\tau)$, $j=1,\cdots,r$ and parameters t_j, $j=1,\cdots,r-1$, such that

(2.6) $$\|x^j(\tau) - \hat{x}^j(\tau)\| \leqq \delta_{01}, \quad \text{for } \tau\in[-1,1],$$

$$(2.7)\qquad |t_j - \hat{t}_j| \leqq \delta_{02},$$

the points $(t_j(\tau),\ x^j(\tau))$, $j=1,\cdots,r$, lie in U for $\tau \in [-1,1]$.

Proof. The result follows immediately from

$$|\hat{t}_j(\tau) - t_j(\tau)| = |\frac{\tau+1}{2}(\hat{t}_j - t_j) + \frac{1-\tau}{2}(\hat{t}_{j-1} - t_{j-1})| \leqq \delta_{02},$$

where $\hat{t}_j(\tau)$ is the linear transformation obtained from (1.11) replacing t_j and t_{j-1} by $\hat{t}_j$ and $\hat{t}_{j-1}$.

By Proposition 3, the system (1.12) can be defined for any $x^j(\tau)$, $j=1,\cdots,r$, and t_j, $j=1,2,\cdots,r-1$, satisfying (2.6) and (2.7).

Since $Y^j(x,t)$, $j=1,\cdots,r$, and $f^j(x,t)$, $j=1,2,\cdots,r-1$, are twice continuously differentiable with respect to x and t in the domain D, there exist following constants $K_1 \sim K_{14}$ such that

$$(2.8)\quad \begin{cases}
\max\limits_{\substack{(x,t)\in U\\ j=1,\cdots,r}} \| Y^j(x,t) \| \leqq K_1, & \max\limits_{\substack{(x,t)\in U\\ j=1,\cdots,r}} \| Y^j_x(x,t) \| \leqq K_2, \\
\max\limits_{(x,t)\in U} \| Y^j_x(x,t) \| \leqq K^j_2, & \max\limits_{\substack{(x,t)\in U\\ j=1,\cdots,r}} \| Y^j_t(x,t) \| \leqq K_3, \\
\max\limits_{\substack{(x,t)\in U\\ j=1,\cdots,r}} \| Y^j_{xx}(x,t) \| \leqq K_4, & \max\limits_{\substack{(x,t)\in U\\ j=1,\cdots,r}} \| Y^j_{xt}(x,t) \| \leqq K_5, \\
\max\limits_{\substack{(x,t)\in U\\ j=1,\cdots,r}} \| Y^j_{tt}(x,t) \| \leqq K_6, & \max\limits_{\substack{(x,t)\in U\\ j=1,\cdots,r}} \| f^j(x,t) \| \leqq K_7, \\
\max\limits_{\substack{(x,t)\in U\\ j=1,\cdots,r}} \| f^j_x(x,t) \| \leqq K_8, & \max\limits_{\substack{(x,t)\in U\\ j=1,\cdots,r}} \| f^j_{xx}(x,t) \| \leqq K_9, \\
\max\limits_{\substack{(x,t)\in U\\ j=1,\cdots,r}} \| f^j_{xt}(x,t) \| \leqq K_{10}, & \max\limits_{\substack{(x,t)\in U\\ j=1,\cdots,r}} \| f^j_{tt}(x,t) \| \leqq K_{11}. \\
\max\limits_{j=1,\cdots,r} \| \frac{d}{d\tau} Y^j_x(x(\tau),\ \hat{t}_j(\tau)) \|_n \leqq K_{12}, &
\end{cases}$$

$$\left\{\begin{array}{l}\max\limits_{j=1,\cdots,r}\left\|\frac{d}{d\tau}Y_t^j(\hat{x}(\tau),\ \hat{t}_j(\tau))\right\|_n \leqq K_{13},\\ \max\limits_{j=1,\cdots,r}\left\|\frac{d}{d\tau}Y^j(\hat{x}(\tau),\ \hat{t}_j(\tau))\right\|_n \leqq K_{14}\ .\end{array}\right.$$

Now we shall provide the properties of the truncated polynomials $\hat{x}_m^j(\tau)$, $j=1,\cdots,r$, of $\hat{x}^j(\tau)$, $j=1,2,\cdots,r$, satisfying (1.12) and (1.13) without proof [see Ref. 2].

$$(2.9)\quad \begin{array}{l}\|\hat{x}^j(\tau) - \hat{x}_m^j(\tau)\|_n \leqq K_{17}^j\frac{\sigma(m)}{m(m-1)}\ ,\\ \|\hat{x}^j(\tau) - \hat{x}_m^j(\tau)\|_q \leqq K_{17}^j\frac{1}{(m+1)m(m-1)},\\ \|\dot{\hat{x}}^j(\tau) - \dot{\hat{x}}_m^j(\tau)\|_n \leqq K_{17}^j[\frac{m+2}{m(m-1)} + \frac{\sigma(m+1)}{m+1}]\ ,\\ \|\dot{\hat{x}}^j(\tau) - \dot{\hat{x}}_m^j(\tau)\|_q \leqq K_{17}^j[\frac{\sqrt{m+2}}{\sqrt{2}m(m-1)} + \frac{1}{(m+2)(m+1)}]\ .\end{array}$$

Here

$$\sigma(m) = \sqrt{2}\left(\sum_{n=m+1}^{\infty}\frac{1}{n^2}\right)^{\frac{1}{2}} \leqq \frac{2}{\sqrt{m}}\ ,$$

$$K_{17}^j = \max_{\substack{(t,x)\in U\\ t\in[t_{j-1},t_j]}} \frac{(t_j-t_{j-1})^3}{2}\Big\|\sum_{k,\ell}\frac{\partial^2 Y^j(x,t)}{\partial x_k\partial x_\ell}Y_k^j(x,t)\, Y_\ell^j(x,t) + [Y_x^j(x,t)^2 + 2\frac{\partial Y_x^j(x,t)}{\partial t}]Y^j(x,t) + Y_x^j(x,t)\frac{\partial Y^j(x,t)}{\partial t} + \frac{\partial^2 Y^j(x,t)}{\partial t^2}\Big\|\ ,$$

where Y_k^j and x_k are respectively a k-th component of Y^j and x. From the first of (2.9), by Proposition 3, there exists a positive integer m_0 such that, for any $m \geqq m_0$, the point $(\hat{x}_m^j(\tau),\ \hat{t}_j(\tau))$ lies in U for $\tau\in[-1,1]$ and $j=1,\cdots,r$, and

$$\Big\{\ \|Y^j(\hat{x}_m^j(\tau),\ \hat{t}_j(\tau)) - Y^j(\hat{x}^j(\tau),\ \hat{t}_j(\tau))\|_n \leqq K_2^j K_{17}^j\frac{\sigma(m)}{m(m-1)},$$

$$(2.10)\quad \begin{cases} \|Y_x^j(\hat{x}_m^j(\tau),\ \hat{t}_j(\tau))-Y_x^j(\hat{x}^j(\tau),\ \hat{t}_j(\tau))\|_n \leqq K_4K_{17}^j\dfrac{\sigma(m)}{m(m-1)}\ , \\ \qquad\qquad j=1,\cdots,r, \\ \|f_x^j(\hat{x}_m^j(1),\ \hat{t}_j)-f_x^j(\hat{x}^j(1),\ \hat{t}_j)\| \leqq K_9K_{17}^j\dfrac{\sigma(m)}{m(m-1)}\ , \\ \|f_t^j(\hat{x}_m^j(1),\ \hat{t}_j)-f_t^j(\hat{x}^j(1),\ \hat{t}_j)\| \leqq K_{10}K_{17}^j\dfrac{\sigma(m)}{m(m-1)}\ , \\ \qquad\qquad j=1,\cdots,r-1. \end{cases}$$

§ 3. Jacobian Matrix of $F^{(m)}(\alpha)$

Let $J_m(\alpha)$ be the Jacobian matrix of $F^{(m)}(\alpha)$. Then we have

$$J_m(\alpha)h = \frac{dF_m(\alpha+\varepsilon h)}{d\varepsilon}\Big/_{\varepsilon=0}\ ,$$

where h is a $(m+1)rn + (r-1)$-vector such that

$$(3.1)\qquad h = \mathrm{col}(h_0^1\cdots h_m^1\cdots h_0^r\cdots h_m^r\eta_1\cdots\eta_{r-1}).$$

Further let β_1 and β_2 be respectively rn-vector and $mrn+r-1$ vector such that

$$(3.2)\qquad \beta_1 = \mathrm{col}(a\ b_1\cdots b_{r-1})\ ,$$

$$(3.3)\qquad \beta_2 = \mathrm{col}(\xi_1^1\cdots\xi_m^1\xi_1^2\cdots\xi_m^2\cdots\xi_1^r\cdots\xi_m^r c_1\cdots c_{r-1}),$$

where a, b_j, ξ_i^k and c_ℓ is respectively the same dimension with z^0, z^j, z_i^k and f^ℓ for each j, k, i, ℓ $(j=1,\cdots,r-1,\ k=1,\cdots,r,\ i=1,\cdots,m,\ \ell=1,\cdots,r)$. Now we shall consider the linear system

$$(3.4)\qquad J_m(\alpha)h = \begin{pmatrix}\beta_1\\ \beta_2\end{pmatrix}\ .$$

If we put

$$(3.5)\quad \begin{cases} x_m^j(\tau) = a_0^j + \sqrt{2}\displaystyle\sum_{n=1}^{m} a_n^jT_n(\tau), \\ y^j(\tau) = h_0^j + \sqrt{2}\displaystyle\sum_{n=1}^{m} h_n^jT_n(\tau), \end{cases}$$

$$(3.6)\qquad \varphi^j(\tau) = \xi_1^j + \sqrt{2}\sum_{n=1}^{m-1} \xi_{n+1}^j T_n(\tau), \qquad j=1,\cdots,r,$$

then, from (1.17) and (3.1), corresponding to (3.4), we have

$$(3.7)\qquad \sum_{i=0}^{1} L_i y^{j_i}((-1)^{i+1}) = a \qquad\qquad (j_0=1,\ j_1=r),$$

$$(3.8)\qquad y^{j+1}(-1) - y^j(1) = b_j, \qquad j=1,\cdots,r-1,$$

$$\begin{aligned}(3.9)\qquad \frac{dy^j}{d\tau} &= P_{m-1}\{Y_x^j(x_m^j(\tau),\ t_j(\tau))y^j(\tau)\}\frac{t_j-t_{j-1}}{2}\\ &+ P_{m-1}Y_t^j(x_m^j(\tau),\ t_j(\tau))\tau_j(\tau)\frac{t_j-t_{j-1}}{2}\\ &+ P_{m-1}Y^j(x_m^j(\tau),\ t_j(\tau))\frac{\tau_j-\tau_{j-1}}{2} - \varphi^j(\tau),\ j=1,\cdots,r,\end{aligned}$$

$$(3.10)\qquad f_t^j(x_m^j(1),\ t_j)y^j(1) + f_t^j(x_m^j(1),\ t_j)\tau_j - c_j = 0,$$

$$j=1,\cdots,r-1.$$

where $\tau_j(\tau)$ is the linear transformation obtained from (1.11) replacing t_j and t_{j-1} by τ_j and τ_{j-1}, and

$$(3.11)\qquad \tau_0 = \tau_r = 0.$$

Let $\hat{x}_m^j(\tau)$ be a function such that

$$\hat{x}_m^j(\tau) = P_m\hat{x}^j(\tau) = \hat{a}_0^j + \sqrt{2}\sum_{n=1}^{m} \hat{a}_n^j T_n(\tau),$$

and we substitute $\hat{x}_m^j(\tau)$, $j=1,\cdots,r$, and $\hat{t}_j$, $j=1,\cdots,r-1$ respectively for $x_m^j(\tau)$, $j=1,\cdots,r$ and t_j, $j=1,\cdots,r-1$ into (3.7) $\sim$(3.10). Then from (3.4), we have

$$(3.12)\qquad J_m(\hat{\alpha})h = \begin{pmatrix}\beta_1\\ \beta_2\end{pmatrix}$$

We shall prove the existence of the inverse of J_m. We put

$$\left(\ g_j(x,\tau) = Y_t^j(x,\hat{t}_j(\tau))\frac{\hat{t}_j-\hat{t}_{j-1}}{2}\,\frac{\tau+1}{2} + \frac{1}{2}Y^j(x,\hat{t}_j(\tau)),\right.$$

$$j=1,2,\cdots,r-1,$$

$$(3.13)\quad \begin{cases} g_r(x,\tau) = 0, \\ q_j(x,\tau) = Y_t^j(x,\hat{t}_j(\tau))\dfrac{\hat{t}_j-\hat{t}_{j-1}}{2}\,\dfrac{1-\tau}{2} - \frac{1}{2}Y^j(x,\hat{t}_j(\tau)) \\ \qquad\qquad j=2,\cdots,r. \\ q_1(x,\tau) = 0, \\ R_{mj}(\tau) = \dfrac{\hat{t}_j-\hat{t}_{j-1}}{2}[P_{m-1}\{Y_x^j(\hat{x}_m^j(\tau),\ \hat{t}_j(\tau))y^j(\tau)\} \\ \qquad - Y_x^j(\hat{x}^j(\tau),\ \hat{t}_j(\tau))y^j(\tau)],\quad j=1,\cdots,r, \\ S_{mj} = f_x^j(\hat{x}_m^j(1),\ \hat{t}_j)y^j(1) + f_t^j(\hat{x}_m^j(1),\ \hat{t}_j)\tau_j \\ \qquad - f_x^j(\hat{x}^j(1),\ \hat{t}_j)y^j(1) - f_t^j(\hat{x}^j(1),\ \hat{t}_j)\tau_j, \\ \qquad\qquad j=1,\cdots,r. \end{cases}$$

Then (3.9) and (3.10) are replaced by

$$(3.14)\quad \frac{dy^j}{d\tau} = Y_x^j(\hat{x}^j(\tau),\hat{t}_j(\tau))y^j(\tau)\frac{\hat{t}_j-\hat{t}_{j-1}}{2} + P_m g_j(\hat{x}_m^j(\tau),\tau)\tau_j$$
$$+ P_{m-1}q_j(\hat{x}_m^j(\tau),\tau)\tau_{j-1} - \varphi^j(\tau) + R_{mj}(\tau),$$

$$(3.15)\quad f_x^j(\hat{x}^j(1),\hat{t}_j)y^j(1) + f_t^j(\hat{x}^j(1),\hat{t}_j)\tau_j - c_j + S_{mj} = 0.$$

The solutions $y^j(\tau)$, $j=1,\cdots,r$ of (3.14) and (3.8) can be obtained by the following propositions.

Proposition 4. For g_j and q_j of (3.13), and $\Phi_j(\tau)$ defined in Proposition 2, we have

$$(3.16)\quad \begin{cases} \|\Phi_j(\tau)\int_{-1}^{\tau}\Phi_j^{-1}(s)P_{m-1}g_j(\hat{x}_m^j(s),s)ds \\ \quad -Y^j(\hat{x}^j(\tau),\ \hat{t}_j(\tau))\frac{\tau+1}{2}\|_q = O(m^{-1}),\ j=1,\cdots,r-1 \\ \|\Phi_j(\tau)\int_{-1}^{\tau}\Phi_j^{-1}(s)P_{m-1}q_j(\hat{x}_m^j(s),s)ds - Y^j(\hat{x}^j(\tau),\hat{t}_j(\tau))\frac{1-\tau}{2} \\ \quad -\Phi_j(\tau)Y^j(\hat{x}^j(-1),\hat{t}_{j-1})\|_q = O(m^{-1}),\ j=2,\cdots,r. \end{cases}$$

Proof. Since $Y^j(\hat{x}^j(\tau),\ \hat{t}_j(\tau))\frac{\tau+1}{2}$ and $Y^i(\hat{x}^i(\tau),\ \hat{t}_i(\tau))\frac{1-\tau}{2}$ is respectively the solution of the system of differential equations

$$\frac{dy}{dt} = Y_x^j(\hat{x}^j(\tau),\ \hat{t}_j(\tau))y\frac{\hat{t}_j-\hat{t}_{j-1}}{2} + g_j(\hat{x}^j(\tau),\tau),$$
$$\text{for } j=1,\cdots,r-1,$$

and

$$\frac{dy}{dt} = Y_x^i(\hat{x}^i(\tau),\ \hat{t}_i(\tau))y\frac{\hat{t}_i-\hat{t}_{i-1}}{2} + q_i(\hat{x}^i(\tau),\tau),$$
$$\text{for } i=2,\cdots,r,$$

(3.16) is proved directly.

Proposition 5. For $R_{mj}(\tau)$ and S_{mj} in (3.13), it holds that

(3.17) $\quad \|R_{mj}(\tau)\|_q \leqq O(m^{-1})(\|y_m^j(\tau)\|_q + \|\eta\| + \|\varphi^j\|_q),$

(3.18) $\quad \|S_{mj}\| \leqq O(m^{-\frac{5}{2}})(\|y_m^j(\tau)\|_q + \|\eta\|),$

where

$$\eta = (\eta_1\cdots\eta_{r-1}).$$

Proof. (3.17) is proved in the same way as the proof with respect to $R_m(\tau)$ of Ref. 2, and (3.18) follows from (2.10).

Proposition 6. Let $y^j(\tau)$, $j=1,\cdots,r$ be solutions obtained successively from the system of differential equations,

(3.19) $\quad \dot{y}^j(\tau) = Y_x^j(\hat{x}^j(\tau),\ \hat{t}_j(\tau))\frac{\hat{t}_j-\hat{t}_{j-1}}{2}y^j(\tau) + \gamma^j(\tau)$

satisfying

$$y^j(-1) = b_{j-1} + y^{j-1}(1), \quad j=1,\cdots,r,$$

with $y^0(1) = 0$. Here $\gamma^j(\tau)$ is a function continuous on $[-1,1]$. Then $y^j(\tau)$ is expressed as

(3.20) $\quad y^j(\tau) = \Phi_j(\tau)\sum_{i=1}^{j}\Phi_{j-1,i}(1)b_{j-1} + \Psi_j(\tau),\ j=1,\cdots,r,$

where

$$\Phi_{j,i}(1) = \Phi_j(1)\cdots\Phi_i(1), \quad i=1,\cdots,j-1,$$

$$\Psi_j(\tau) = \Phi_j(\tau)\int_{-1}^{\tau}\Phi_j^{-1}(s)\gamma^j(s)ds$$
$$+\ \Phi_j(\tau)\sum_{i=1}^{j-1}\Phi_{j-1,i}(1)\int_{-1}^{1}\Phi_i^{-1}(s)\gamma^i(s)ds,$$
$$j=1,\cdots,r.$$

Further we have

$$\|\Psi_j(1)\| \leqq \sum_{i=1}^{j} K_{15i}\|\gamma^i\|_q,$$
$$\|\Psi_j(\tau)\|_q \leqq \sum_{i=1}^{j} K_{16i}\|\gamma^i\|_q\ ,$$

where

$$K_{15i} = \|\Phi_{j,i}(1)\|\sqrt{\int_0^{\pi}\Pi\|\Phi_j^{-1}(\cos\ \theta)\|^2\sin^2\theta d\theta},$$

$$K_{16i} = [\int_0^{\pi}\|\Phi_j(\cos\ \theta)\|^2 d\theta]^{1/2}\|\Phi_{j-1,i}(1)\|\sqrt{\int_0^{\pi}\|\Phi_j^{-1}(\cos\ \theta)\|^2\sin^2\theta d\theta},$$
$$i=1,\cdots,j.$$

Proof. (3.20) is proved directly by mathematical induction and (3.21) follows from (1.15).

Now from (1.3) follows

(3.22) $\quad Y^j(\hat{x}(\hat{t}_j),\ \hat{t}_j) = Y^{j+1}(\hat{x}(\hat{t}_j),\ \hat{t}_j), \quad j=1,\cdots,r-1,$

Then, by (3.22) and Proposition 4$\sim$Proposition 6, the solutions $y^j(\tau)$, $j=1,\cdots,r$ of (3.14) and (3.8) satisfying the initial condition

$$y^1(-1) = b_0$$

is expressed as

(3.23) $\quad y^j(\tau) = \Phi_j(\tau)\sum_{i=1}^{j}\Phi_{j-1,i}(1)b_{i-1} + Y^j(\tau)\frac{\tau+1}{2}\eta_j$

$$+\ Y^j(\tau)\frac{1-\tau}{2}\eta_{j-1} + \Psi_j(\tau) + Q_{mj}(\tau),\ j=1,\cdots,r,$$

where

$$Y^j(\tau) = Y^j(\hat{x}^j(\tau), \hat{t}_j(\tau))$$

$$\bar{\Psi}_j(\tau) = -\Phi_j(\tau)\Big(\int_{-1}^{\tau}\Phi_j^{-1}(s)\varphi^j(s)ds + \sum_{i=1}^{j-1}\Phi_{j-1,i}(1)\int_{-1}^{1}\Phi_i^{-1}(s)\varphi^i(s)ds\Big),$$

$$\|Q_{mj}(\tau)\| \leqq O(m^{-1})(\|\tau\|+|\varphi^j\|_q+\|y^j\|_q).$$

Substituting (3.23) and (3.11) into (3.7) and (3.15), we have

(3.24) $$L_0 b_0 + L_1\Big[\sum_{i=1}^{r}\Phi_{r,i}(1)b_{i-1}+\bar{\Psi}_r(1)+Q_{mr}(1)\Big] = a,$$

(3.25) $$f_x^j(\hat{x}^j(1),\hat{t}_j)\Big(\sum_{i=1}^{j}\Phi_{j,i}(1)b_{i-1}+Y^j(1)\tau_j+\bar{\Psi}_j(1)+Q_{mj}(1)\Big) + f_t^j(\hat{x}^j(1),\hat{t}_j)\tau_j+S_{mj} = c_j \qquad j=1,\cdots,r-1.$$

Thus, from the isolatedness and the transversality condition, b_0 and τ_j are determined as follows :

(3.26) $$b_0 = G^{-1}\{a-L_1(\sum_{i=2}^{r}\Phi_{r,i}(1)b_{i-1}+\bar{\Psi}_r(1)+Q_{mr}(1))\}$$

(3.27) $$\tau_j = A_j^{-1}[c_j-f_x^j\cdot(\bar{\Psi}_j(1)+Q_{mj}(1)+\Phi_{j,1}(1)G^{-1}a) + \sum_{i=2}^{r} H_{3i}(j)b_{i-1}+f_x^j\cdot\Phi_{j,1}(1)G^{-1}L_1(\bar{\Psi}_r(1)+Q_{mr}(1))-S_{mj}],$$
$$j=2,\cdots,r-1,$$

where

$$f_x^j = f_x^j(\hat{x}^j(1),\ \hat{t}_j), \qquad j=2,\cdots,r-1$$

$$H_{3i}(j) = \begin{cases} f_x^j\Phi_{j,1}(1)G^{-1}L_1\Phi_{r,i}(1) - f_x^j\Phi_{j,i}(1), & j=2,\cdots,r-1, \\ f_x^j\Phi_{j,1}(1)G^{-1}L_1\Phi_{r,i}(1), & i=j+1,\cdots,r. \end{cases}$$

Moreover, by the substitution (3.26) and (3.27) into (3.23), we have

$$(3.28)\quad y^j(\tau) = (\Phi_j(\tau)-Y^j(\tau)\frac{\tau+1}{2}A_j^{-1}f_x^j\cdot\Phi_j(1)-Y^j(\tau)\frac{1-\tau}{2}A_{j-1}^{-1}\cdot f_x^{j-1})$$

$$\cdot\Phi_{j-1,1}(1)G^{-1}a+\sum_{i=2}^{r}H_{4i}(j)b_{i-1}+\Phi_j(\tau)\Phi_{j-1,1}(1)G^{-1}\Psi_r(1)$$

$$+\ Y^j(\tau)\frac{\tau+1}{2}A_j^{-1}[c_j-f_x^j\cdot\Psi_j(1)+f_x^j\Phi(t_j)G^{-1}L_1\Psi_r(1)]$$

$$+\ Y^j(\tau)\frac{1-\tau}{2}A_{j-1}^{-1}[c_{j-1}-f_x^{j-1}\cdot\Psi_{j-1}(1)G^{-1}L_1\Psi_r(1)]+\Psi_j(\tau)$$

$$+P_{mj}(\tau),$$

where

$$H_{4i}(j) = \begin{cases} -\Phi_j(\tau)\Phi_{j-1,1}(1)G^{-1}L_1\Phi_{r,i}(1)+\Phi_i(\tau)\Phi_{j-1,i}(1) \\ +Y^j(\tau)\frac{\tau+1}{2}A_j^{-1}H_{3i}(j)+Y^j(\tau)\frac{1-\tau}{2}A_{j-1}^{-1}H_{3i}(j-1),\ i=2,\cdots,j, \\ -\Phi_j(\tau)\Phi_{j-1,1}(1)G^{-1}L_1\Phi_{r,i}(1)+Y^j(\tau)\frac{\tau+1}{2}A_j^{-1}H_{3i}(j-1) \\ +Y^j(\tau)\frac{1-\tau}{2}A_{j-1}^{-1}H_{3i}(j-1),\quad i=j+1,\cdots,r. \end{cases}$$

$$\| P_{mj}(\tau)\|_q \leqq O(m^{-1})(\|\tau\| + \|\varphi\|_q+\|y\|_q),$$

with $\|\varphi\|_q = (\sum_{j=1}^{r}\|\varphi^j\|_q^2)^{1/2}$ and $\|y\|_q = (\sum_{j=1}^{r}\|y^j\|_q)^{1/2}$.

If we put, in (3.12), $\beta_1 = 0$, $\beta_2 = 0$, that is, $a = 0$, $b_i = 0$, $c_i = 0$, $\varphi^j(\tau) \equiv 0$, $i=1,\cdots,r-1$, $j=1,\cdots,r$, then by (3.27) and (3.28), follows $\tau = 0$, $y^j(\tau) \equiv 0$, $j=1,\cdots,r$, that is, $h = 0$. Therefore it readily follows that, for sufficiently large m,

$$\det J_m(\hat{\alpha}) \neq 0,$$

or, there exists the inverse of $J_m(\hat{\alpha})$.

Now the linear formulas of $F^{(m)}(\alpha)$ are $z^0(\alpha)$ and $z^j(\alpha)$, $j=1,\cdots,r-1$, and the nonlinear formulas are z_i^j, $i=1,\cdots,m$, $j=1,\cdots,r$, and f^j, $j=1,\cdots,r-1$. Then the number of linear formulas of $F^{(m)}(\alpha)$ is rn. We shall evaluate $\|J_m^{-1}\begin{pmatrix}0&0\\0&E\end{pmatrix}\|$.

and $\|J_m^{-1}\begin{pmatrix}E & 0\\ 0 & 0\end{pmatrix}\|$, where $\begin{pmatrix}0 & 0\\ 0 & E\end{pmatrix}$ and $\begin{pmatrix}E & 0\\ 0 & 0\end{pmatrix}$ are the matrices of the same form as J_m, E is a unit $(rnm+r-1)\times(rnm+r-1)$-matrix in the former and a unit $rn\times rn$-matrix in the latter, and O's are zero matrices. From (3.2) and (3.3), the dimension of β_1 and β_2 is respectively the number of linear formulas and nonlinear formulas of $F^{(m)}(\alpha)$. Then the norms of $J_m^{-1}\begin{pmatrix}0 & 0\\ 0 & E\end{pmatrix}$ and $J_m^{-1}\begin{pmatrix}E & 0\\ 0 & 0\end{pmatrix}$ are

(3.29)
$$\|J_m^{-1}\begin{pmatrix}0 & 0\\ 0 & E\end{pmatrix}\| = \sup_{(\beta_1,\beta_2)} \frac{\|J_m^{-1}\begin{pmatrix}0 & 0\\ 0 & E\end{pmatrix}\begin{pmatrix}\beta_1\\ \beta_2\end{pmatrix}\|}{\sqrt{\|\beta_1\|^2+\|\beta_2\|^2}} = \sup_{\beta_2}\frac{\|J_m^{-1}\begin{pmatrix}0\\ \beta_2\end{pmatrix}\|}{\|\beta_2\|},$$
$$\|J_m^{-1}\begin{pmatrix}E & 0\\ 0 & 0\end{pmatrix}\| = \sup_{\beta_1}\frac{\|J_m^{-1}\begin{pmatrix}\beta_1\\ 0\end{pmatrix}\|}{\|\beta_1\|}.$$

By the substitution $\beta_1 = 0$ into (3.27) and (3.28), we have, for sufficiently large m,

$$[\|y\|_q^2+\|\eta\|^2]^{1/2} \leqq M_1^{1/2}\,[\sum_{j=1}^{r-1} c_j^2+\|\varphi\|_q^2]^{1/2},$$

or

(3.30) $$\|J_m^{-1}\begin{pmatrix}0 & 0\\ 0 & E\end{pmatrix}\| \leqq M_1.$$

Here M_1 is a constant such that

$$M_1 \geqq \frac{(\max_j|A_j^{-1}|)^2+\sum_{j=1}^{r-1} A_j^{-2}\sum_{i=1}^{r} H_{1i}^2(j)+(2K_1\max_{j=1,\cdots r-1}|A_j^{-1}|)^2}{1-O(m^{-1})}$$

$$\frac{+\sum_{j=1}^{r}\sum_{i=1}^{r} H_{2i}(j)+O(m^{-1})}{},$$

where

$$H_{1i}(j)\begin{cases}(\|f_x^j\|+\|f_x^j\Phi_{j,1}(1)G^{-1}L_1\|)K_{15i}, & i=1,\cdots,j,\\ \|f_x^j\Phi_{j,1}(1)G^{-1}L_1\|K_{15i}, & i=j+1,\cdots,r,\end{cases}$$

$$H_{2i}(j) = \begin{cases} \|\Phi_j(\tau)\Phi_{j-1,1}(1)G^{-1}L_1\|_n K_{15i} + K_1(|A_j^{-1}|H_{1i}(j) \\ +|A_{j-1}^{-1}|H_{1i}(j-1)) + K_{16i}, \qquad \text{for } i=1,\cdots,j, \\ \|\Phi_j(\tau)\Phi_{j-1,1}(1)G^{-1}L_1\|_n K_{15i} + K_1(|A_j^{-1}|H_{1i}(j) \\ +|A_{j-1}^{-1}|H_{1i}(j-1)) \qquad \text{for } i=j+1,\cdots,r,\ j=1,\cdots,r. \end{cases}$$

$\|J_m^{-1}\begin{pmatrix} E & 0 \\ 0 & 0 \end{pmatrix}\|$ is also evaluated, by the substitution $\beta_2 = 0$ into (3.27) and (3.28), as follows :

$$[\|\eta\|^2 + \|y\|_q^2]^{\frac{1}{2}} \leqq M_2(\|a\|^2 + \sum_{i=2}^{r}\|b_{j-1}\|^2)^{\frac{1}{2}}$$

or

$$(3.31) \qquad \|J_m^{-1}\begin{pmatrix} E & 0 \\ 0 & 0 \end{pmatrix}\| \leqq M_2,$$

where M_2 is a constant such that

$$M_2 \geqq [\sum_{j=1}^{r-1}\|A_j^{-1}f_x^j\cdot\Phi_{j,1}(1)G^{-1}\|^2 + \sum_{j=1}^{r-1}A_j^{-2}\sum_{i=2}^{r}H_{3i}^2(j) + \sum_{j=1}^{r}\ \sum_{i=2}^{r}H_{4i}^2(j)$$
$$+\sum_{j=1}^{r}\{\|\Phi_j(\tau)\|_n + K_1|A_j^{-1}|\,\|f_x^j\cdot\Phi_j(1)\| + K_1|A_{j-1}^{-1}|K_8\}^2\ \cdot$$
$$\cdot\|\Phi_{j,1}(1)G^{-1}\|^2]^{\frac{1}{2}}/(1-O(m^{-1})).$$

Now we put

$$J_m^2(\alpha) = [0,E]J_m(\alpha),$$

where 0 is a zero $(rnm+r-1)\times rn$ matrix and E a $(rnm+r-1)\times(rnm+r-1)$ unit matrix. Thus $J_m^2(\alpha)$ is a Jacobian matrix corresponding to the nonlinear formulas of $F^{(m)}(\alpha)$. Let

$$\alpha' = \mathrm{col}(a_0^{1'}\cdots a_m^{1'}\,a_0^{2'}\cdots a_m^{2'}\cdots a_0^{r'}\cdots a_m^{r'}\,t_1'\cdots t_{r-1}')$$

and

$$\alpha'' = \mathrm{col}(a_0^{1''}\cdots a_m^{1''}\,a_0^{2''}\cdots a_m^{2''}\cdots a_0^{r''}\cdots a_m^{r''}\,t_1''\cdots t_{r-1}'')$$

be arbitrary $(m+1)rn+(r-1)$-vectors such that

$$|t_j'-\hat{t}_j| \leqq \delta_{02}, \quad |t_j''-\hat{t}_j| \leqq \delta_{02}, \qquad j=1,2,\cdots,r-1,$$

and both $x^{j\prime}(\tau) = a_0^{j\prime}+\sqrt{2}\sum_{n=1}^{m} a_n^{j\prime} T_n(\tau)$ and $x^{j\prime\prime}(\tau) = a_0^{j\prime\prime} +\sqrt{2}\sum_{n=1}^{m} a_n^{j\prime\prime} T_n(\tau)$ satisfy

$$\|x^{j\prime}(\tau)-\hat{x}^j(\tau)\|_n \leqq \delta_{01} \text{ and } \|x^{j\prime\prime}(\tau)-\hat{x}^j(\tau)\|_n \leqq \delta_{01},$$

$$\text{for } j=1,\cdots,r.$$

Here δ_{01} and δ_{02} be numbers in (2.6) and (2.7). Then, by the similar calculations of $\|J_m(\alpha')-J_m(\alpha'')\|$ in the reference 2, we have the following important inequality in the proof of Theorem.

$$\|J_m^2(\alpha')-J_m^2(\alpha'')\| \leqq (2m+1)M_3\|\alpha'-\alpha''\| \tag{3.32}$$

with $M_3 = \{K_9^2+2K_{10}^2+K_{11}^2+K_4^2+8K_5^2+2K_2^2+16(K_6^2+K_3^2)+8K_5K_2+32K_6K_3\}^{\frac{1}{2}}$.

§ 4. Proof of Theorem

By Proposition 3 and the first of (2.9), the point $(\hat{x}_m^j(\tau),\hat{t}_j(\tau))$ ($\tau \in [-1,1]$, $j=1,\cdots,r$) lies in U for any $m \geqq m_0$ provided m_0 is sufficiently large. For such m, let us put

$$r_0 = \sum_{i=0}^{1} L_i \hat{x}_m^{j\,i}((-1)^{i+1}), \quad r_1^j = f(\hat{x}_m^j(1), \hat{t}_j),$$

$$R_m^j = \frac{d\hat{x}_m^j(\tau)}{d\tau} - P_{m-1}Y^j(\hat{x}_m^j(\tau), \hat{t}_j(\tau))\frac{\hat{t}_j-\hat{t}_{j-1}}{2},$$

$$\text{for } \tau \in [-1,1],\ j=1,\cdots,r.$$

Then, in the same way as §3-4 of Ref. 2, we have, for some constant K,

$$\|F^{(m)}(\hat{\alpha})\| = \sqrt{\|r_0\|^2+\sum_j \|R_n^j\|_q^2+\sum_j \|r_1^j\|^2} \leqq Km^{-\frac{3}{2}}, \tag{4.1}$$

$$m \geqq m_1 \geqq m_0 .$$

(4.1) expresses that $\alpha = \hat{\alpha}$ is an approximate solution of the determining E_q.(1.17). From the first of (2.9) follows

(4.2) $\|\hat{x}_m^j(\tau) - \hat{x}^j(\tau)\|_n \leqq K_{17}\frac{\sigma(m)}{m(m-1)}$,

with $K_{17} = \max_{j=1,\cdots,r} K_{17}^j$. For K_{17}, let us denote by Ω_m the set

$$\Omega_m = \{\alpha=\mathrm{col}(\xi^1\cdots\xi^r t_1 t_2\cdots t_{r-1}) : \|\xi^j-\hat{\xi}^j\| \leqq \frac{1}{\sqrt{2m+1}}[\delta_{01} - \frac{K_{17}\sigma(m)}{m(m-1)}],\ j=1,2,\cdots,r,\ |t_j-\hat{t}_j| \leqq \delta_{02},\ j=1,2,\cdots,r-1\},$$

where

$\xi^j = \mathrm{col}(a_0^j,a_1^j,\cdots,a_m^j)$, $\hat{\xi}^j = \mathrm{col}(\hat{a}_0^j,\hat{a}_1^j,\cdots,\hat{a}_m^j)$, $j=1,2,\cdots,r$.

For $x_m^j(\tau) = a_0^j+\sqrt{2}\sum_{n=1}^{m} a_n^j T_n(\tau)$, $j=1,2,\cdots,r$ and t_j, $j=1,2,\cdots,r-1$, such that

(4.3) $\alpha = (a_0^1,a_1^1,\cdots,a_m^1,\cdots,a_0^r,a_1^r,\cdots,a_m^r,\ t_1,\cdots,t_{r-1}) \in \Omega_m$,

we have, by the third of (1.16),

$$\|x_m^j(\tau)-\hat{x}_m^j(\tau)\|_n \leqq \delta_{01} - \frac{K_{17}\sigma(m)}{m(m-1)},\ j=1,\cdots,r,$$

$$|t_j-\hat{t}_j| \leqq \delta_{02},\ j=1,2,\cdots,r-1.$$

Then, by (4.2) and Proposition 3, the point $(x_m^j(\tau),\ t_j(\tau))$, $(\tau\in[-1,1],\ j=1,\cdots,r)$ corresponding (4.3) included in U. This means that $F^{(m)}(\alpha)$ is defined in Ω_m. For an arbitrary number $\kappa < 1$, and the numbers M_1,M_2 and M_3 in (3.30)～(3.32), we put

$$\delta_1 = \min.[\frac{\kappa}{M_3 M_1},\ \delta_{01} - \frac{K_{17}\sigma(m_1)}{m_1(m_1-1)},\ \delta_{02}]$$

and take $m_2 \geqq m_1$ so that

$$\frac{(M_1+M_2)K}{1-\kappa} m^{-\frac{3}{2}} < \frac{\delta_1}{2m+1},\ \text{for any } m \geqq m_2 .$$

Let us now take δ_m such that

$$\frac{(M_1+M_2)K}{1-\kappa} m^{-\frac{3}{2}} \leqq \delta_m \leqq \frac{\delta_1}{2m+1}$$

and consider the region

$$\Omega_{\delta_m} = \{\alpha : \|\alpha-\hat{\alpha}\| \leqq \delta_m\}.$$

Then

(4.4) $$\Omega_{\delta_m} \subset \Omega_m$$

and further, for any $\alpha \in \Omega_{\delta_m}$ and $m \geqq m_2$,

(4.5) $$\| J_m^2(\alpha) - J_m^2(\hat{\alpha})\| \leqq \frac{\kappa}{M_1}$$

(4.6) $$\frac{(M_1+M_2)\| F^{(m)}(\hat{\alpha})\|}{1-\kappa} \leqq \delta_m.$$

Thus, from (4.4)$\sim$(4.6), and Proposition 1, we see that the determining Eq. (1.17) has the unique solution $\alpha = \bar{\alpha}$ in Ω_{δ_m}. This proves the existence of the approximate solution in the form of the connection of Chebyshev series tied at switching times. From (1.25) in Proposition 1 and (4.2), the convergence (1.21) is proved in the same way as §3.5 in Ref. 2.

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