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We are sorry to inform that our ex-Chairman, Prof. Minoru Urabe passed away on September 5, 1975.

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On the Stability Condition of Difference Schemes for Initial Value Problems

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1. Introduction

The stability condition for the scheme of difference equations with constant coefficients with periodic boundary condition is proved by P.D. Lax and R.D. Richtmyer(5), to be equivalent to the uniform boundedness of the norm (see §2) $|G^n(\Delta t,k)|$ in $D=\{0<\Delta t<\tau, 1\leqq n\leqq T/\Delta t,k\}$. Here $k=(k_1,\cdots,k_d)$, $k_i=2\pi r_i/L_i$, $r_i=0,\pm1,\pm2,\cdots$, d is the number of spatial variables and L_i are the periods. $G(\Delta t,k)$ is the amplification matrix introduced by the Fourier transformation of the difference equation, and its dimension m is equal to the number of dependent variables.

J. von Neumann's necessary condition on the eigenvalues λ of G is $|\lambda| \leqq 1+K\Delta t$, where K is a constant in D.

T. Kato (3) showed that, roughly speaking, if the larger eigenvalues of G have simple elementary divisors and if either (1) the eigenvalues are separated from each other by a fixed amount, or (2) the associated normalized eigenvectors have the Gram determinant (see §2) larger than a fixed number, then the von Neumann condition is sufficient.

H.O. Kreiss [4] obtained four necessary and sufficient conditions, but his conditions are often difficult to apply to the practical matrix calculation.

M.L. Buchanan [1] found the necessary and sufficient condition for the upper triangular matrix form, to which $G(\Delta t,k)$ can be transformed by a unitary matrix into

$$(1.1) \qquad A(\Delta t,k) = \begin{pmatrix} \lambda_1(\Delta t,k) & a_{12}(\Delta t,k) & \cdots & a_{1,m}(\Delta t,k) \\ 0 & \lambda_2(\Delta t,k) & \cdots & a_{2,m}(\Delta t,k) \\ & & \cdots & \cdots \\ 0 \cdots 0 & & \lambda_{m-1}(\Delta t,k) & a_{m-1,m}(\Delta t,k) \\ 0 \cdots & \cdots & 0 & \lambda_m(\Delta t,k) \end{pmatrix}$$

with the diagonal elements nested with a nesting constant R(i.e. $|\lambda_i-\lambda_j| \leqq R|\lambda_r-\lambda_s|$ for all ranges with $1 \leqq r \leqq i \leqq j \leqq s \leqq m$).

Her condition for the stability is that there exist constants K and L such that in D

$$|\lambda_i| \leqq 1 + K\Delta t, \qquad 1 \leqq i \leqq m,$$

and*

$$(1.2) \qquad \frac{|a_{ij}|}{\max(\Delta t/T,\ 1-|\lambda_j|,|\lambda_i-\lambda_j|)} \leqq L,\ 1\leqq i<j\leqq m\,.$$

The lemma (1) of T.Kato is deduced from her result, but the lemma (2) is not.

The equivalence theorem of P.D. Lax and R.D Richtmyer is for the limit $\Delta t\to 0$. The actual calculation is practised for a finite Δt. Richtmyer and Morton showed the examples in which the stability condition is satisfied and the difference calculation runs practically unstable even in early time steps. Hence they imposed a severer condition on the maximum eigenvalue $|\lambda|_{max}$ in D for a finite Δt for practical stability as a modified von Neumann condition [8].

In this paper we take the eigenspace into consideration. In §2 we show that if $|G|$ is bounded and its spectral sets are separated each other by a fixed amount, then the Gram determinant of the matrix of the normalized basis of the subspaces for these sets can be made larger than a positive constant. Hence Kato's condition (2) is deduced from his condition (1). And Buchanan's criterion (1.2) is necessary only for the set of such eigenvalues $\lambda_i(\Delta t,k(\Delta t))$ in D, that converge to a multiple eigenvalue for $k=k(\Delta t)$ in the limit $\Delta t\to 0$.

In §3 we show that occasionally the norm increases with n for some eigenvalues λ, i.e. $|G^n|\sim n^h|\lambda|^{n-1}|G|$, where $h\geqq 1$. This would

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* Her result is misquoted by Richtmyer and Morton [7] and V. Thomée [9].

cause the practical instability in the early time steps, rather than $|G^n| \approx |\lambda|^n_{max}$, in the case $|\lambda|_{max} \sim 1$.

If $|\lambda|<1$, it decreases to 0 for sufficiently large n, then the practical instability will disappear, unless there is a large eigenvalue $|\lambda|_{max}>1$ in D.

In §4 two examples of the practical instability are discussed. One of them indicated by H.Takami (Fig.2) is that, $|\lambda|=1$ in D and the stability condition is not satisfied, but the modified von Neumann condition is satisfied. We show in this example that for the eigenvalue with index 2, G~4 and $|G^n|$~4n, which causes the practical instability, where $|\lambda|_{max}=1$.

In another example, one in Richitmyer and Morton's book[8], the modified von Neumann condition is satisfied and yet the condition for the eigenvalue to be of index 2 is satisfied. Therefore, the practical instability occurs in the early time steps.

2. Estimation of $|G^n|$ by $|G^n|_x$

Before stating our results, we recall some relevant notions of the spectral theory and matrix representation of linear operators (see Dunford and Schwartz [2], T.Kato [3] and van der Waerden [10]). Let G be a linear operator in unitary space X of dimension m(dim X=m<∞), $\lambda_1, \cdots, \lambda_s$ ($s \leq m$) be the distinct eigenvalues of G, and the set $\sigma=\{\lambda_1,\cdots,\lambda_s\}$ be the spectrum of G. The set of all $u \in X$ such that $(G-\lambda_i)^n u=0$ for some integer n forms an algebric eigenspace $X(\lambda_i)$ of X. The smallest n for which this is true for all $u \in X(\lambda_i)$ is called the index of λ_i and is denoted by η_i. A vector $u \neq 0$ is an eigenvector of G for λ_i if $(G-\lambda_i)u=0$. Any subset

$\sigma' \subseteq \sigma$ is called a spectral set, and the set $X(\sigma')$ of all $u \in X$ such that $\prod_{\lambda_i \in \sigma'} (G-\lambda_i)^n u_i = 0$ for some integer n is an invariant subspace for G. For two separated sets σ' and $\sigma''(\sigma' \cap \sigma''=0)$, the spectral distance of σ' and σ'' is defined by $\text{dist}(\sigma',\sigma'')= \min_{\substack{\lambda' \in \sigma' \\ \lambda'' \in \sigma''}} |\lambda'-\lambda''|$.

Let $\{x_1, \cdots, x_m\}$ be an orthonormal basis in the unitary space X. Then a vector $u = \sum_1^m a_i x_i \in X$ is represented by the column matrix

$$u = \begin{bmatrix} a_1 \\ a_2 \\ \cdot \\ \cdot \\ \cdot \\ a_m \end{bmatrix}.$$

G can be represented by an m×m matrix as a linear transformation of the column matrix u into Gu. In the following, u, v, ··· indicate column vectors. The inner product of u and $v(= \sum_1^m b_i x_i)$ is $(u,v)= \sum_1^m \bar{a}_i b_i$, and the norm of vector u is $|u|=(u,u)^{1/2}$. Further the norm of matrix G is defined by $|G|= \max_{0 \neq u \in X} |Gu|/|u|$ and the norm in invariant subspace $X'=X(\sigma')$ is defined by $|G|_{X'}= \max_{0 \neq u \in X'} |Gu|/|u|$.

Let $\{u_1', \cdots, u_{m'}'\}$, be a basis of any invariant subspace $X'=X(\sigma')$ and $U'=[u_1', \cdots, u'_{m'}]$ be an m×m' matrix of which the columns are the basis, then G is reduced to an m'×m' matrix G' such that GU'=U'G'. Further let U=[U', U", ···] be the union of such matrices U', U", ···, then G is reduced to a direct sum such as $GU=U \cdot (G' \oplus G'' \oplus \cdots)$.

The Gram determinant of U' is a determinant of the product of conjugate transposed matrix U'* and U', i.e. det[U'*.U'], and is a square of the volume of parallel polyhedron spanned by the basis $\{u_1', \cdots, u'_{m'}\}$.

We consider the initial value problem with constant coefficients and with simple boundary condition, so that the Fourier integral representation of the solution can be used. If there are p dependent variables in d-dimensional space, then the differential equation is given as

$$\frac{\partial}{\partial t} u(x,t) = P\left(\frac{\partial}{\partial x}\right) u(x,t), \tag{2.1}$$

where u is a vector with p components and x is a vector with d components. $P(\frac{\partial}{\partial x})$ is a p×p matrix whose elements are polynomials of $\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_d}$.

If the solution of the initial value problem is expressed as

$$u(x,t) = (2\pi)^{-d/2} \int dk\, \hat{u}(k,t) e^{i(k,x)},$$

then (2.1) becomes

$$\frac{\partial}{\partial t} \hat{u}(k,t) = P(ik)\hat{u}(k,t).$$

Hence the solution is

$$u(k,t) = e^{tP(ik)} u(k,0). \tag{2.2}$$

Now let us start our statements. Let G be a linear operator in a unitary space X, and σ' and σ'' be separated spectral sets of G, then $X'=X(\sigma')$ and $X''=X(\sigma'')$ are invariant subspaces of G. Let $\dim X'=m'$, $\dim X''=m''$, and $\eta' = \sum_{\lambda_i \in \sigma'} \eta_i$, $\eta'' = \sum_{\lambda_i \in \sigma''} \eta_i$ be the sums of the indices.

The first lemma reads:

Lemma 1. Let $u' \in X'$ and $u'' \in X''$ be vectors such that $|u'|=|u''|=1$, then the following inequality holds for the Gram determinant:

$$\det\{[u',u'']^* \cdot [u',u'']\} \geq \left(\frac{\operatorname{dist}(\sigma',\sigma'')}{|2G|}\right)^{2M_1}, \tag{2.3}$$

where $M_1=\min(\eta'm'', \eta''m')$.

Proof. Let $u''\in X''$ be decomposed such that

(2.4) $$u'' = cu'+w, \quad (u',w)=0,$$

where the coefficient c is given by

$$(u',u'')=c(u',u')=c,$$

Since $u'\in X'$, we have

$$\prod_{\lambda_i\in\sigma'}(G-\lambda_i)^{\eta_i}u'=0.$$

Therefore

$$\left|\prod_{\lambda_i\in\sigma'}(G-\lambda_i)^{\eta_i}u''\right|=\left|\prod_{\lambda_i\in\sigma'}(G-\lambda_i)^{\eta_i}w\right| \leqq \prod_{\lambda_i\in\sigma'}(|G|+|\lambda_i|)^{\eta_i}|w|$$

(2.5) $$\leqq |2G|^{\eta'}|w|, \quad \eta'=\sum_{\lambda_i\in\sigma}\eta_i .$$

The operator $\prod_{\lambda_i\in\sigma'}(G-\lambda_i)^{\eta_i}$ is regular in the invariant subspace $X''=X(\sigma'')$.

It follows that

$$|u''|=\left|\prod_{\lambda_i\in\sigma'}(G-\lambda\ )^{-\eta_i}\cdot\prod_{\lambda_i\in\sigma'}(G-\lambda\ )^{\eta_i}u''\right|$$

(2.6) $$\leqq\left|\prod_{\lambda_i\in\sigma'}(G-\lambda_i)^{-\eta_i}\right|_{X''}\cdot\left|\prod_{\lambda_i\in\sigma'}(G-\lambda_i)^{\eta_i}u''\right|$$

$$\leqq\left|\prod_{\lambda_i\in\sigma'}(G-\lambda_i)^{-\eta_i}\right|\cdot|2G|^{\eta'}|w|.$$

On the other hand, Kato's lemma (see[3]) says for any linear operator T in a unitary space X with the inverse T^{-1} that

$$|T^{-1}|\leqq|T|^{m-1}/|\det T|, \qquad m=\dim X.$$

For an invariant subspace $X''\in X$, the lemma is that

(2.7) $$|T^{-1}|_{X''}\leqq|T|_{X''}^{m''-1}/|\det T_{X''}|, \qquad m''=\dim X'',$$

where $|T|_{X''}$ is the norm in X'' and $T_{X''}$ is the restriction of T on X'' for $T = \prod_{\lambda_i \in \sigma'} (G-\lambda_i)^{\eta_i}$.

Then (2.7) becomes

$$|(G-\lambda_i)^{-1}|_{X''} \leqq |2G|^{m''-1}/\mathrm{dist}(\sigma',\sigma'')^{m''}.$$

Since

$$\Big|\prod_{\lambda_i \in \sigma'} (G-\lambda_i)^{\eta_i}\Big|_{X''} \leqq \Big|\prod_{\lambda_i \in \sigma'} (G-\lambda_i)^{\eta_i}\Big| \leqq |2G|^{\eta'},$$

we have for $\lambda_i \in \sigma'$

$$|\det(G-\lambda_i)_{X''}| = \prod_{\lambda_j \in \sigma'} |\lambda_j - \lambda_i| \geqq \mathrm{dist}(\sigma',\sigma'')^{m''}.$$

Substituting the above relation into (2.6), we have

$$(2.8) \qquad |w| \geqq \left(\frac{\mathrm{dist}(\sigma',\sigma'')}{|2G|}\right)^{\eta' m''} |u''|.$$

Then by (2.3) the Gram determinant becomes

$$\begin{aligned} \det\{[u',u'']^* \cdot [u',u'']\} &= \det\{[u',cu'+w]^* \cdot [u',cu'+w]\} \\ &= \det\{[u',w]^* \cdot [u',w]\} \end{aligned}$$

$$(2.9) \qquad = \begin{vmatrix} |u'|^2 & 0 \\ 0 & |w|^2 \end{vmatrix} \geq \frac{\mathrm{dist}(\sigma',\sigma'')}{|2G|}^{2\eta' m''}$$

$$|u'| = 1.$$

In (2.9), X' and X'' can be exchanged and hence the lemma 1 is proved.

The second lemma reads:

Lemma 2. Let normalized basis of X', X'' be $\{v'_1,\cdots,v'_{m'}\}$ and $\{v''_1,\cdots,v''_{m''}\}$, their matrices be $V'=[v'_1,\cdots,v'_{m'}]$, $V''=[v''_1,\cdots,v''_{m''}]$ then the following inequality holds for the Gram determinant:

$$\det\{[V',V'']^*\cdot[V',V'']\}\geq\left(\frac{\mathrm{dist}(\sigma',\sigma'')}{|2G|}\right)^{2M_2}\cdot\det\{V'^*\cdot V'\}\cdot\det\{V''^*\cdot V''\},$$

where

$$M_2 = \min(\eta' m''^2, \eta'' m'^2).$$

Proof. We choose a normalized basis $\{u''_i \in X'',\ i=1,\cdots,m''\}$ so that the following decomposition holds:

(2.10) $\quad u''_i = u'_i + w_i,\quad u'_i \in X',\quad w_i \perp X'$ and $(w_i,w_j)=0$ for $i\neq j$.

Let the matrix of their vectors be

$$U''=[u''_1\cdots\cdot u''_{m''}],\quad U'=[u'_1\cdots\cdot u'_{m''}] \text{ and } W=[w_1\cdots w_{m''}],$$

then

$$U'' = U' + W.$$

Let the matrix of any normalized basis $\{v'_i \in X',\ i=1,\cdots,m''\}$ be

$$V' = [v'_1\cdots\cdots v'_{m'}].$$

Then the Gram determinant is given as follows:

$$\det\{[V',U'']^*\cdot[V',U'']\} = \det\{[V',U'+W]^*\cdot[V',U'+W]\}$$

(2.11)
$$= \det\{[V',W]^*\cdot[V',W]\} = \begin{vmatrix} V'^*\cdot V' & 0 \\ 0 & \begin{pmatrix} |w_1|^2 & & \\ & \ddots & \\ & & |w_{m''}|^2 \end{pmatrix} \end{vmatrix}$$

$$= \det\{V'^*\cdot V'\}\cdot\prod_{i=1}^{m''}|w_i|^2,$$

by making use of (2.10). Let the matrix of any normalized basis $\{v''_i \in X'',\ i=1,\cdots,m''\}$ be $V''=[v''_1,\ldots,v''_{m''}]$.

Since U'' is also the matrix of normalized basis in X'', there exists a regular matrix S'' such that

$$(2.12) \qquad V'' = U'' S'',$$

Then

$$[V',V''] = [V',U''] \cdot \begin{pmatrix} I_{m'} & 0 \\ 0 & S'' \end{pmatrix},$$

where $I_{m'}$ is a unit matrix of dimension m'. Therefore, it follows for the Gram determinant that

$$(2.13) \qquad \begin{aligned} \det\{[V',V'']^* \cdot [V',V'']\} &= \det\{[V',U'']^* \cdot [V',U'']\} \cdot \det\{S''^* \cdot S''\} \\ &= \det\{V'^* \cdot V'\} \cdot \det\{S''^* \cdot S''\} \cdot \prod_{i=1}^{m''} |w_i|^2, \end{aligned}$$

by (2.11) and (2.12), and for the Gram determinant of (2.12) that

$$(2.14) \qquad \begin{aligned} \det\{V''^* \cdot V''\} &= \det\{U''^* \cdot U''\} \cdot \det\{S''^* \cdot S''\} \\ &\leqq \det\{S''^* \cdot S''\}, \end{aligned}$$

since U'' is a matrix of normalized basis so that $\det\{U''^* \cdot U''\} \leqq 1$.

Substitution of (2.14) into (2.13) yields

$$(2.15) \qquad \det\{[V',V'']^* \cdot [V',V'']\} \geqq \det\{V'^* \cdot V'\} \cdot \det\{V''^* \cdot V''\} \prod_{i=1}^{m''} |w_i|^2,$$

where

$$\prod_{i=1}^{m''} |w_i|^2 \geqq \left(\frac{\operatorname{dist}(\sigma',\sigma'')}{|2G|}\right)^{2M_1 m''}$$

from lemma 1. Since X', X'' can be exchanged in (2.15), Lemma 2 is proved.

Hence Kato's condition (1) in §1 is deduced from his condition (2).

Lemma 3. Let T and U be m-dimensional square matrices, rows or columns of U being normalized, and let

$$\det\{U^* \cdot U\} \geqq \Delta^2 > 0.$$

Then

$$|[U^{-1}\ T\ \ U]_{ij}| \leqq \frac{m^2}{\Delta} \max_{i,j}|T_{ij}|,$$

$$|[U\ \ T\ U^{-1}]_{ij}| \leqq \frac{m^2}{\Delta} \max_{i,j}|T_{ij}|.$$

This lemma is easily proved by Cramér's rule. Further, the following lemma can be deduced from lemma 2 and lemma 3.

Lemma 4. Let the spectral set σ of a linear operator G in a unitary space X be decomposed into two subsets

$$\sigma = \{\sigma', \sigma''\},$$

with

$$\text{dist}\,\{\sigma', \sigma''\} \geqq \rho > 0,$$

and let X' and X'' be subspaces for σ' and σ'', then

$$m^3\left(\frac{|2G|}{\rho}\right)^{-M_2} \max(|G^n|_{X'}, |G^n|_{X''}) \leqq |G^n|$$

$$\leqq m^3\left(\frac{|2G|}{\rho}\right)^{+M_2} \cdot \max(|G^n|_{X'}, |G^n|_{X''}).$$

In the difference scheme of an initial value problem the dimension m of the amplification matrix G is nearly equal to the number of dependent variables. Elements of G are rational expressions of time step Δt, space differences Δx_j and $\exp[ik_j\Delta x_j]$, where $j=1, 2, \cdots, d$, and d is the number of space variables [5].

For the difference scheme, a functional relation $\Delta x_j = \Delta x_j(\Delta t)$ is assumed such that $\Delta x_j \to 0$ as $\Delta t \to 0$. Hence we may take the amplification matrix for $G(\Delta t, k)$ in D.

Take any functional relation $k = k(\Delta t)$ in D. If $\sigma(k(\Delta t))$ is the maximum spectral set of eigenvalues which converge to a multiple eigenvalue and if the invariant space $X' = X(\sigma'(k(\Delta t)))$ is defined in $0 < \Delta t < \tau$, then the following theorem is derived from lemma 4.

Theorem. The necessary and sufficient condition for $|G^n(\Delta t,k)|$ to be uniformly bounded in D is that there exists a constant M

$$|G^n(k(\Delta t))|_{X'} < M,$$

in D and for all $k(\Delta t)$, where $X'=X(\sigma'(k(\Delta t)))$ is such a subspace that $\sigma'(k(\Delta t))$ is a spectral set of which elements converge to a multiple eigenvalue as $\Delta t \to 0$.

Particularly when all eigenvalues have index 1 in the limit $\Delta t \to 0$, $|G^n(k(\Delta t))|_{X'}$ becomes $|\lambda^n(k(\Delta t))|$ and this condition coincides with the von Neumann condition. In this case the von Neumann condition is sufficient as well as necessary.

3. Practical stability condition for the finite Δt.

The stability condition is a condition in the limit of $\Delta t \to 0$. Richtmyer and Morton [8] showed the various examples which satisfy the stability condition, but their calculations were practically unstable for a finite Δt. They take as a modified von Neumann condition the restriction

$$|\lambda(\Delta t,k)| \leqq e^{r\Delta t}, \tag{3.1}$$

where $r = \max \operatorname{Re}\xi$, ξ is the eigenvalue of $P(ik)$ in (2.2) for all k and all eigenvalues for each k. Although the practical instability often depends on the initial condition, we consider only the practical instability caused from the amplification of the vector by G^n. The power of the eigenvalue $|\lambda^n|_{max}$ is not always adequate to use for the norm $|G^n|$.

In fact the square norm $|G|^2$ equals the maximum eigenvalue of G^*G, but $|G|^2$ is not suitable to deal with the iterated norm $|G^n|$. Sometimes it happens that $|\lambda|_{max}$ is near 1. In this case

$|\lambda|^n_{max}$ grows slowly for small n, but the calculation becomes practically unstable in the early time steps, although $|\lambda|^n_{max}$ varies slowly for small n.

In the following we estimate $|G^n|$ in terms of its eigenspace, and we show that $|G^n|$ grows sometimes as $n^h|\lambda|^{n-1}|G|$ for small n, where $h \geq 1$. We shall let the elements of G remain constant and increase n.

(1) Let λ', λ'' be distinct eigenvalues of G, and u' and u'' be normalized eigenvectors for respective eigenvalues. For the eigenvectors lemma 1 is simplified as

$$(3.2) \qquad |G| \geq \frac{|\lambda' - \lambda''|}{2(\det\{[u',u'']^* \cdot [u',u'']\})^{1/2}} .$$

Then

$$|G^n| \geq \frac{|\lambda'^n - \lambda''^n|}{2(\det\{[u',u'']^* \cdot [u',u'']\})^{1/2}} ,$$

where

$$(3.3) \qquad \begin{aligned} |\lambda'^n - \lambda''^n| &\geq ||\lambda'|-|\lambda''||\cdot|\lambda''|^{n-1}\cdot|(1-|\lambda'/\lambda''|^n)/(1-|\lambda'/\lambda''|)| \\ &\geq ||\lambda'|-|\lambda''||\cdot|\lambda''|^{n-1}\cdot n \qquad \text{for } |\lambda'|>|\lambda''|. \end{aligned}$$

For $|\lambda'|<1$, $|G^n|$ decreases to 0 for sufficiently large n.

Concerning the limiting case $\lambda''\to\lambda'$ in (3.2) we deal with the eigenvalue of which the index is larger than unity.

(2) Let the restriction of G on the irreducible eigenspace $X'=X(\lambda')$ be

$$(3.4) \qquad A' = \begin{pmatrix} \lambda' & a_{12} & & \cdots\cdots & a_{1m'} \\ 0 & \lambda' & a_{23} & & a_{2m'} \\ & & & \cdots\cdots\cdots & \\ & & & \lambda' & a_{m'-1,\ m'} \\ & & & 0 & \lambda' \end{pmatrix}$$

for an orthonormal basis of X'. Then it is valid that $|G^n| \geqq |G^n|_{x'} = |A'^n|$. We decompose (3.4) into two parts such that

$$(3.5) \qquad A' = \lambda' I_{m'} + A'',$$

where $I_{m'}$ is an unit matrix of dimension m' and A'' is the remainder matrix for which we can show $(A'')^{m'}=0$. Using (3.4), we have

$$(3.6) \qquad A'^n = \lambda'^n I_{m'} + n\lambda'^{n-1}A'' + \cdots + \binom{n}{h}\lambda'^{n-h}A''^h + \cdots + \binom{n}{m'-1}\lambda'^{n-m'+1}A''^{m'-1}.$$

Since (i,i+1) elements are zero in A''^2, $A''^3, \cdots$, we have

$$(3.7) \qquad |A'^n|_{x'} \geqq n|\lambda'|^{n-1} \cdot \max_i |a_{i,i+1}|,$$

where $a_{i,\ i+1} \neq 0$ for $i=1,\cdots, m'-2$, as X' is irreducible.

Next let

$$\max_h \left| \frac{A''^h}{h! \cdot \lambda'} \right|^{1/h} = B,$$

then from (3.6) we have

$$|A'^n|_{x'} < |\lambda'| \frac{n(nB)^{m'} - 1}{nB-1} < 2(nB)^{m'-1}|\lambda'|^n, \quad \text{for } nB \geqq 2.$$

In the case of $|\lambda'|=1-c$, where c is a small positive number, the above power attains to its maximum at $n=(m'-1)/c$, and then decreases to 0 as n increases.

Thus even if the stability condition for $\Delta t \to 0$ is satisfied, some practical instability may occur, when the distance between the normalized eigenvectors is small, or when there is a large coefficient in the upper triangular matrix for orthonormal basis of the irreducible eigenspace. If the absolute value of the eigenvalue in question is smaller than 1, this practical instability may disappear for sufficiently large n.

4. Examples of the practical instability.

We apply the above considerations to the following examples. In the first example of the numerical solution for the wave equation by H. Takami (private commumication. Fig.2) the absolute value of the eigenvalue is 1, and therefore the stability condition is not satisfied, but the modified von Neumann condition is satisfied. The eigenvalue with index 2 is -1 and the upper triangular element of (3.6) is 4, and hence $|G| \sim 4$.

In the second example by Richtmyer and Morton about the wave equation coupled with heat flow (Fig.3.see [8]), an instability occurs from the first step and grows more and more with n. In this case the stability condition is satisfied but the modified von Neumann condition is not. The maximum absolute eigenvalue is 1.7. We use the same example to see the effect of existence of the eigenvalue with index 2 of the amplification matrix. The modified von Neumann condition is necessary for the condition that no eigenvalue of index 2 should appear, although both conditions almost coincide numerically (Fig.1).

Example 1. The wave equation is $\partial^2u/\partial t^2=\partial^2u/\partial x^2$, or equivalently

$$(4.1)\qquad \begin{cases} \dfrac{\partial v}{\partial t} = \dfrac{\partial w}{\partial x}, \\[2ex] \dfrac{\partial w}{\partial t} = \dfrac{\partial v}{\partial x}, \end{cases}$$

where $v(x,t)=\partial u/\partial t$, $w(x,t)=\partial u/\partial x$. According to the scheme of Courant-Friedrichs-Lewy, we set

$$(4.2)\qquad \begin{cases} v_j^n = v(j\Delta x,\ n\Delta t), \\[2ex] w_j^n = w((j+\frac{1}{2})\Delta x,\ (n+\frac{1}{2})\Delta t). \end{cases}$$

The finite difference equation corresponding to (4.1) is

$$(4.3)\qquad \begin{cases} v_j^{n+1} = v_j^n + \nu(w_j^n - w_{j-1}^n), \\[2ex] w_j^{n+1} = w_j^n + \nu(v_{j+1}^{n+1} - v_j^{n+1}) \\[2ex] \qquad\ = w_j^n + \nu(v_{j+1}^n - v_j^n) + \nu^2(w_{j+1}^n - 2w_j^n + w_{j-1}^n), \end{cases}$$

where $\nu=\Delta t/\Delta x$. The amplification matrix is

$$(4.4)\qquad G(k,\Delta t) = \begin{bmatrix} 1 & 2i\nu e^{-i\omega/2}\sin(\omega/2) \\ 2i\nu e^{i\omega/2}\sin(\omega/2) & 1-4\nu^2\sin^2(\omega/2) \end{bmatrix},$$

where $\omega=k\Delta x$, $k=2\pi r/L$, r being an integer.

The eigenvalues are

$$\lambda=(1-2\nu^2\sin^2(\omega/2)) \pm 2\nu\sin(\omega/2)\sqrt{\nu^2\sin^2(\omega/2)-1}.$$

Therefore, $|\lambda|_{max}>1$ for $|\nu\sin(\omega/2)|\geq 1$.

Let $\nu=1$, then the matrix (4.4) is diagonalized as

$$(4.5) \qquad \Lambda(\omega) = \begin{bmatrix} e^{-i\omega} & 0 \\ 0 & e^{i\omega} \end{bmatrix},$$

by the matrix U of the normalized eigenvectors:

$$(4.6) \qquad U(\omega) = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & e^{-i\omega/2} \\ -1 & e^{i\omega/2} \end{bmatrix}.$$

Eigenvalues of (4.5) tend to multiple eigenvalues -1 as ω tends to π, and the Gram determinant of U tends to 0.

According to the statements mentioned in §3, the matrix of an orthonormal basis for (4.4) is

$$(4.7) \qquad V = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

and G is transformed by the unitary matrix (4.7) into

$$(4.8) \qquad A(\omega) = \begin{bmatrix} e^{-i\omega} & 4\sin^2(\omega/2) \\ 0 & e^{i\omega} \end{bmatrix}.$$

The stability condition is not satisfied due to Buchanan's criterion, nevertheless the modified von Neumann condition (3.1) is satisfied. Further

$$(4.9) \qquad A^n(\omega) = \begin{bmatrix} e^{-in\omega} & 2\tan(\omega/2)\cdot\sin(n\omega) \\ 0 & e^{in\omega} \end{bmatrix}.$$

For $\omega=\pi$, the above matrices become

$$(4.10) \qquad A(\pi) = \begin{bmatrix} -1 & 4 \\ 0 & -1 \end{bmatrix},$$

and

$$(4.11) \qquad A^n(\pi) = \begin{bmatrix} (-1)^n & (-1)^{n-1}4n \\ 0 & (-1)^n \end{bmatrix}.$$

The multiple eigenvalue $\lambda(\pi)=-1$ has index 2, and by (4.11) $|A^n|$ grows approximately as 4n, while $|\lambda|^n=1$.

In this case ($\nu=1$), the finite difference equation (4.3) has the general solution:

$$(4.12)\quad \begin{cases} v(x,t)=f(x+t)-g(x-t) \\ w(x,t)=f(x+t)+g(x-t) \end{cases} \sim \begin{cases} v_j^n=F(j+n)-G(j-n)\ , \\ w_j^n=F(j+n+1)+G(j-n)\ , \end{cases}$$

with arbitrary functions $F(j)=f(j\Delta t)$ and $G(j)=g(j\Delta t)$. Note that the solution (4.12) are the same as that of the differential equation (4.1) restricted on the mesh points.

Now we give some examples to see the behavior of the solution of the practically unstable finite difference equation (4.3) in the case of $\nu=1$. Let us take an initial condition:

$$(4.13)\quad \begin{cases} v^0 = \begin{cases} 1\ , & x \leq 0\ , \\ 0\ , & x > 0\ , \end{cases} \\ w^0 = 0, \end{cases}$$

then we obtain the following solution:

$$(4.14)\quad \begin{cases} v(x,t) = \begin{cases} 1\ , & x \leq -t\ , \\ 1/2\ , & -t<x\leq t\ , \\ 0\ , & t < x\ , \end{cases} \\ w(x,t) = \begin{cases} 0\ , & x \leq -t\ , \\ -1\ , & -t<x\leq t\ , \\ 0\ , & t < x\ , \end{cases} \end{cases}$$

so that the initial condition for the finite difference equation (4.3) is

$$
(4.15)\qquad \begin{cases} v_j^0 = \begin{cases} 1, & j \le 0, \\ 0, & j < 0, \end{cases} \\[2ex] w_j^0 = \begin{cases} -{}^1/_2, & j = 0, \\ 0, & j \neq 0, \end{cases} \end{cases}
$$

and its solution is the restriction of (4.14) on the mesh points.

In stead of (4.15), we take an initial condition:

$$
(4.16)\qquad \begin{cases} v_j^0 = \begin{cases} 1, & j \le 0, \\ 0, & j > 0, \end{cases} \\[2ex] w_j^0 = 0. \end{cases}
$$

This is equivalent to take additional functions $\bar{v}$ and $\bar{w}$, the initial condition of which is

$$
(4.17)\begin{cases} \bar{v}_j^0 = 0, \\ \bar{w}_j^0 = \begin{cases} +{}^1/_2, & j = 0, \\ 0, & j \neq 0. \end{cases} \end{cases}
$$

Its solution is

$$
\bar{v}_j^n = E(n+j)-E(n-j) = \begin{cases} (-1)^{n+j-1}/2, & -n<j\le n, \\ 0, & \text{else}, \end{cases}
$$

$$
\bar{w}_j^n = E(n+j+1)+E(n-j) = \begin{cases} (-1)^{n+j}/2, & -n\le j\le n, \\ 0, & \text{else}, \end{cases}
$$

where

$$
E(j) = \begin{cases} 0, & j \leqq 0, \\ (-1)^{j-1}/2, & j > 0. \end{cases}
$$

To introduce periodic condition, we take the convolutions of the above solution and a periodic function of which the mean value is zero : e.g.

$$P(j) = \begin{cases} -1, & j=0, \\ +1, & j=N_0 : 0<N_0<N, \\ 0, & \text{else}, \end{cases}$$

where N is the period of the functions. Since the equation is linear, the convolutions

$$Pv_j^n \equiv \sum_{\ell=-\infty}^{\infty} \sum_{k=0}^{N-1} P(k+\ell N)v_{j-k}^n, \qquad \text{etc.}$$

are also the solution. It can be shown that Pv_j^n etc. are well defined and periodic with the period N.

Consider, for example, the error terms $P\bar{v}_j^n$ and $P\bar{w}_j^n$. If both N and N_0 are even, amplitudes of these convolutions grow infinitely as $n\to\infty$. This corresponds to the infinite growth of the iterated amplification matrix $|G^n|$ when $\omega=\pi$. In fact, the infinitely growing component of $P\bar{v}_j^n$ is the one proportional to $(-1)^{j+n}$. When either N_0 or $N-N_0$ is odd, no solution grows indefinitely because N is fixed and no Fourier component grows indefinitely. Anyway, the amplitude of $P\bar{v}_j^n$ is the same order of magnitude as that of Pv_j^n itself, and hence the profiles of Pv_j^n and $P(v_j^n + \bar{v}_j^n)$ are quite different.

Aside from the wave equation, it is difficult to find an initial condition with no loss of accuracy and with no growing components. In practical usage, components for large $k(=N\omega)$ are not important, and practical instabilities occur at such large k's, since the components for small k behave approximately like solutions of the differential equation by virtue of the consistency property of finite difference equation. Therefore, we can avoid the instability at the expense of accuracy; that is, suitably mollifying the initial conditions we apply the same difference equation to get rather smooth solution which contains small components with large k's.

For example, we start by the initial conditions which contain the convolution of the given initial values and the smoothing function $\varphi(j)=\binom{J}{j}2^{-J}$, with an integer $J\geqq 0$. Spatial shift by $J/2$ is brought about, and its correction is necessary.

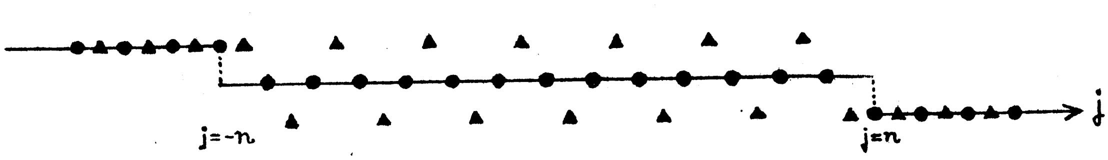
$$\varphi(j) = 2^{-J}\binom{J}{j} \xrightarrow{\text{Fourier Transform}} (\cos(k\pi/N))^J \qquad |k|\leq N/2 ,$$

$$Pv_j^0 * \varphi = 2^{-J} \sum_{s=0}^{J} \binom{J}{s} Pv_{j-s}^0 \longrightarrow (\cos(k\pi/N))^J \widetilde{Pv}^0(k) .$$

Let us apply this scheme to the solution $P\bar{v}_j^n$, $P\bar{w}_j^n$ $(J=1)$.

$$P\bar{v}_j^n * \varphi = \begin{cases} 1/4 , & j=-n+1 , \\ -1/4 , & j=n+1 , \\ 0 , & \text{else} . \end{cases}$$

The following figure will be useful to understand the above result



—— the exact solution of differential equation

▲ the solution of finite difference equation with the initial condition (4.16)

● the smoothing solution of ▲ with correction shift by $-1/2$.

Note that, in the case of wave equation, the convolutions with $\frac{1}{2}(\frac{1}{j})$ give the exact solution for $v_j^0=0$, and they give the solution shifted by $\Delta t/2$ on the space axis and by $\Delta t/2$ on the time axis for $w_j^0=0$. These facts are deduced from the general solution (4.12).

Next let us take the following initial condition:

$$(4.18)\qquad \begin{cases} v^0 = 2f(x) = \begin{cases} 1-x^2, & |x|\le 1, \\ 0, & |x|>1, \end{cases} \\ w^0 = 0 . \end{cases}$$

Its solution is

$$\begin{cases} v = f(x+t)+f(x-t) \\ w = f(x+t)-f(x-t) \end{cases} \sim \begin{cases} v_j^n = F(j+n)+F(j-n), \\ W_j^n = F(j+n+1)-F(j-n) . \end{cases}$$

Let us take the initial condition for the finite difference equation corresponding to (4.18) such as

$$(4.19)\qquad \begin{cases} v_j^0 = 2F(j) = \begin{cases} 1-(j\Delta t)^2, & |j\Delta t|\le 1, \\ 0, & \text{otherwise}, \end{cases} \\ w_j^0 = F(j+1)-F(j) = \begin{cases} -(j+1/2)\Delta t^2, & -1\le j\Delta t\le 1-\Delta t, \\ 0, & \text{otherwise}, \end{cases} \end{cases}$$

where $\Delta t=1/M$ (M: a positive integer). We obtain the exact solution for (4.1) and (4.18) as the solution of the finite difference eqs. (4.3) and (4.19).

If, instead, we start with the initial condition :

$$\begin{cases} v_j^0 = 2F(j), \\ w_j^0 = 0 , \end{cases}$$

we get the solution :

$$\begin{cases} v_j^n = F(j+n)+F(j-n)+\delta_j^n , \\ w_j^n = F(j+n+1)-F(j-n)+\varepsilon_j^n , \end{cases}$$

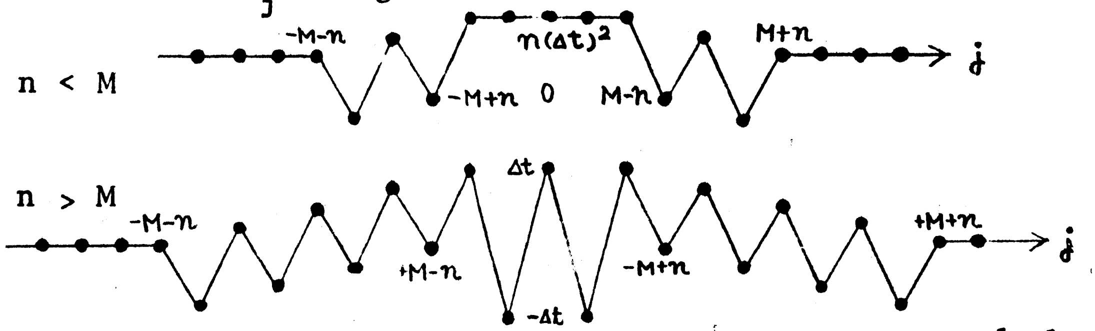
with additional error terms :

$$\begin{cases} \delta_j^n = \varphi(j+n)-\varphi(j-n) , \\ \varepsilon_j^n = \varphi(j+n+1)+\varphi(j-n) , \end{cases}$$

where

$$\varphi(j) = \begin{cases} 0 , & j\Delta t \leq -1 , \\ (j\Delta t+(-1)^{j+M})\Delta t , & -1 \leq j\Delta t \leq +1 , \\ (-1)^{j-M}\Delta t , & j\Delta t \geq 1 , \end{cases}$$

the profile of δ_j^n is given as follows :



In this case also, the mollification by the function $\frac{1}{2}\binom{1}{j}$ gives the exact solution.

Example 2. The wave equation coupled with heat flow.

This stability condition is discussed by Richtmyer and Morton. The amplification matrix for their scheme is

$$(4.20)\qquad G = \begin{bmatrix} 1 & i\nu s & -i(\gamma-1)\nu s \\ i\nu s & 1-\nu^2 s^2 & (\gamma-1)\nu^2 s^2 \\ \dfrac{-i\nu s}{\mu s^2+1} & \dfrac{\nu^2 s^2}{\mu s^2+1} & \dfrac{-(\gamma-1)\nu^2 s^2+1}{\mu s^2+1} \end{bmatrix},$$

where $\gamma=c_p/c_v$, σ=thermal conductivity/c_v, $s=2\sin(\frac{k}{2}\Delta x)$, $k=2\pi r/L$, r is an integer, $\nu=c\Delta t/\Delta x$ and $\mu=\sigma\Delta t/(\Delta x)^2$.
The stability condition [6] is

$$(4.21)\qquad \nu = c\Delta t/\Delta x < 1.$$

Though the above stability condition is satisfied, the calculation of the propagation of a simple jump shows sometimes the unstable profile from the first step. The modified von Neumann condition is $|\lambda|\leq 1$, and leads to the practical stability condition

$$(4.22)\qquad \nu < \sqrt{(1+2\mu)/(\gamma+2\mu)}\ .$$

Fig.3 shows the unstable profile in an early time step for

$$(4.23)\qquad \gamma=3.0,\quad \nu=(\sqrt{3}+2)/5,\quad \text{and } \nu/\mu=5/3,$$

where the stability condition (4.21) is satisfied but the modified von Neumann condition is not, and the instability grows more and more with n.

Now, we shall show that even if the stability condition $c\Delta t/\Delta x<1$ and the modified von Neumann condition $\nu<\sqrt{((1+2\mu)/(\gamma+2\mu))}$ are satisfied at the same time, the solution can be practically unstable. First we shall look for the condition for the matrix G of (4.20) is reduced to the diagonal form

$$G \sim \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{bmatrix}.$$

Then the rank of $G-\lambda_1$ is 1, the necessary and sufficient condition for this is obtained from (4.20) as

$$\lambda_1=1 \quad \text{and} \quad s=0 \quad \text{or} \quad \nu=0 .$$

For this case the rank is 0 and G is a unit matrix. Hence, it is found that the double eigenvalue has the index 2 when G has two equal and one distinct eigenvalues. We shall look for the condition for G to have double eigenvalues.

The characteristic equation for (4.20) is

$$g(\lambda)=(\mu s^2+1)\lambda^3+(\mu\nu^2 s^2-(2\mu-\gamma\nu^2)s^2-3)\lambda^2+((\mu-\gamma\nu^2)s^2+3)\lambda-1=0, \tag{4.24}$$

and since

$$\frac{dg}{d\lambda} = 0$$

for the double eigenvalue, we get

$$f(\lambda,\kappa)=(\gamma\kappa+(\lambda-1)(\lambda+1))(\kappa^{-1}(\lambda-1)+\gamma(\lambda+1))-(\lambda-1)(\lambda+2)^2=0, \tag{4.25}$$

where

$$\kappa = \nu^2/\mu \qquad \text{and}$$

$$\nu^2 s^2 = \frac{-(\lambda-1)((1+\gamma\kappa)\lambda-(1-\gamma\kappa))}{\lambda(\lambda+2)} . \tag{4.26}$$

The double root of the cubic equation (4.24) with real coefficients is real, and from (4.25) we see that the double root of (4.13) is in the interval

$$-1 < \lambda \leqq -\frac{2}{\sqrt{\gamma}+1}$$

It is not difficult to see that the eigenvalue of the greatest modulus is near -1 in the complex plane, and that it moves monotonically to the double root of the modulus less than 1 when we vary the parameters (μ, ν, s) continuously, and if $s=2\sin(k\pi/N)\to 0$, the corresponding components become stable in any sense. Theis means that the modified von Neumann condition $|\lambda|<1$ is necessary for G to have no double eigenvalue of index 2.

Therefore, we investigate the case of $\nu/\mu=\alpha$, where α is the lower bound of ν/μ to have the double eigenvalue λ for each fixed ν. In this case the component corresponding to the double eigenvalue λ is one with s=-2 (i.e. $k\Delta x=\pi$), that is, the component is proportional to $(-1)^{n-j}|\lambda|^n$, where λ is noted to be real. This suggests that the mollification used in the first example is also effective. Since the general solution or any other elementary expression for the solution is not found, some of our numerical results are given in Fig 4a) and b).

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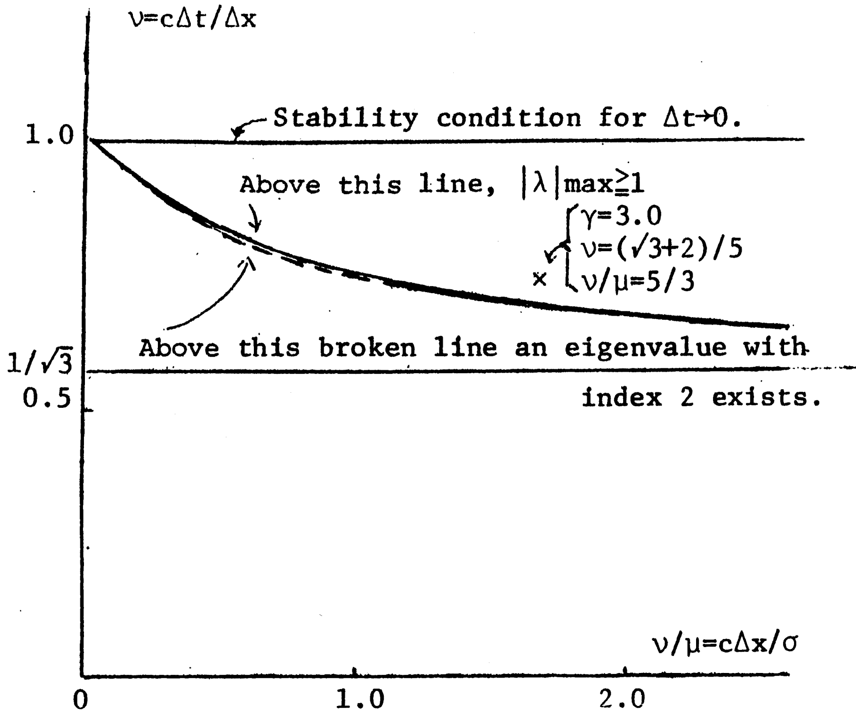

Fig.1. Stability diagram for coupled sound and heat flow for $\gamma=3$, between $\nu=c\Delta t/\Delta x$ and $\mu/\nu=c\Delta x/\sigma$. The stability condition is satisfied below the curve. Two lower curves approach still more as γ tends to 1.

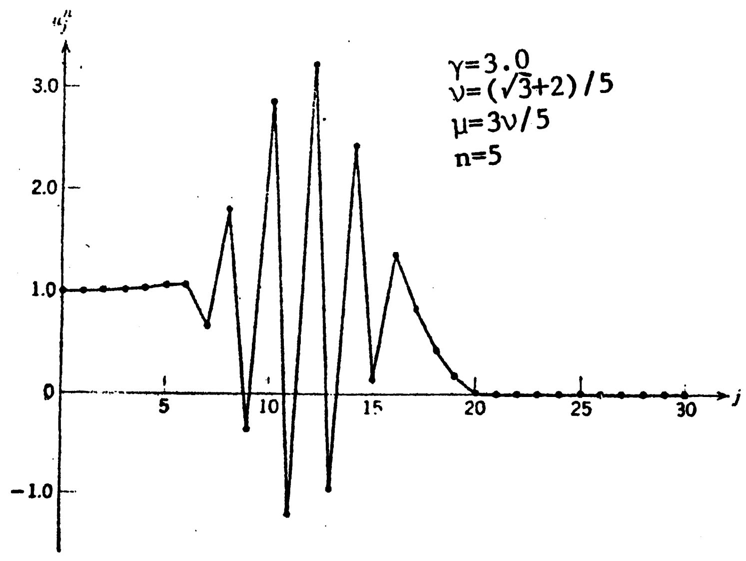

Fig.3. Calculated profile after 5 cycles, in a run in which the modified von Neumann condition was violated, although $c\Delta t/\Delta x<1$ (From Richtmyer and Morton[8]).

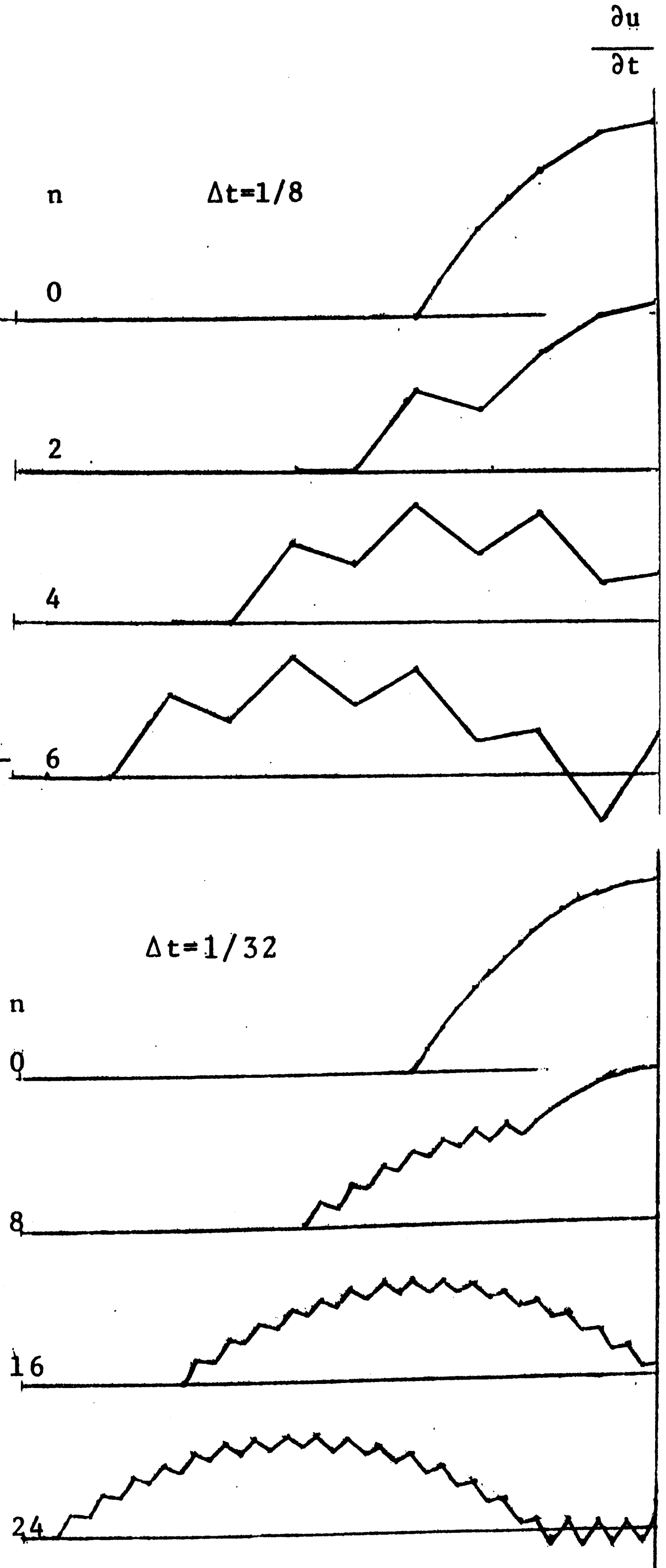

Fig.2. Profiles of the solution of wave equation in example 1, in §4, for $\nu=\Delta t/\Delta x=1$ (By H.Takami)

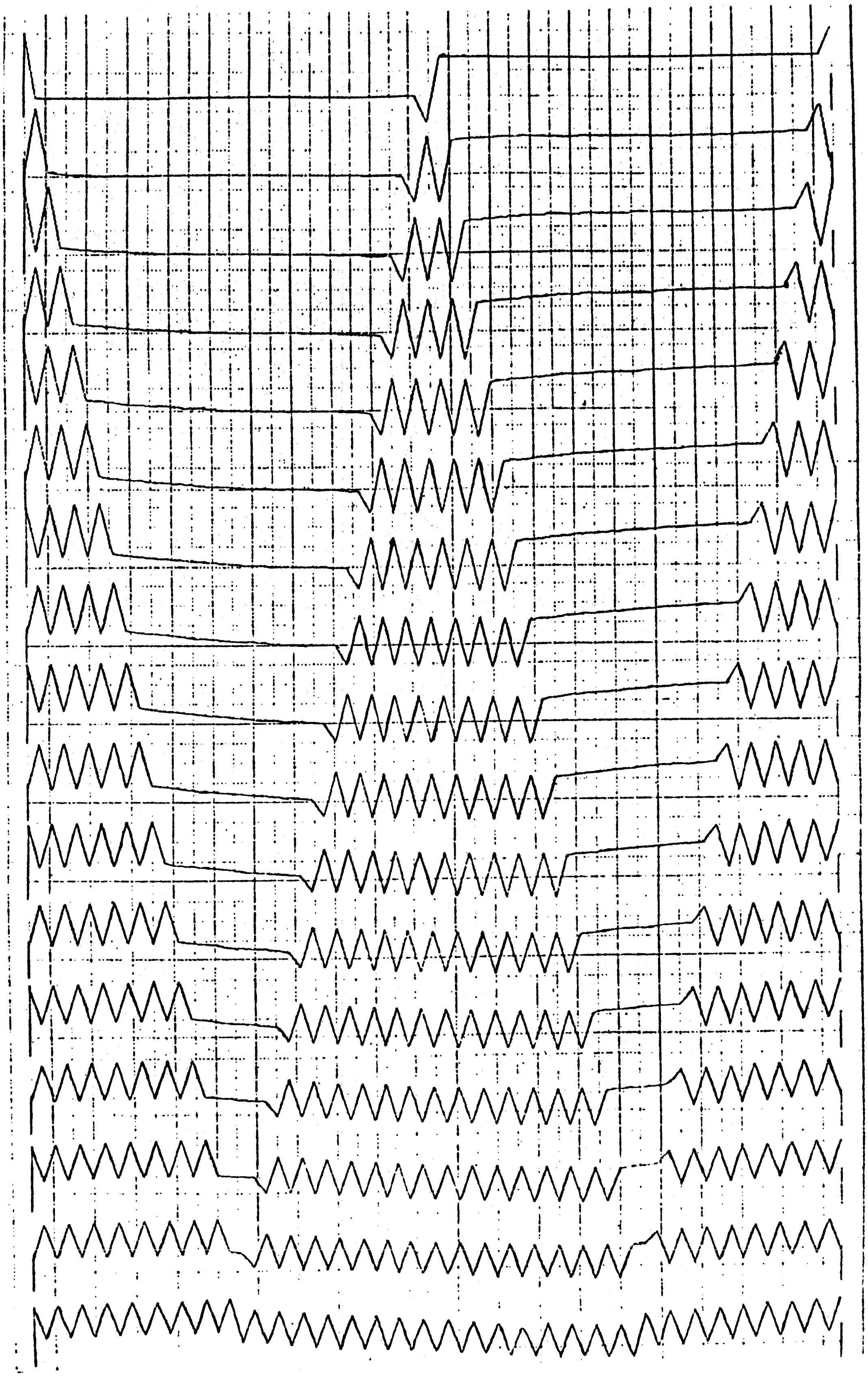

Fig.4 a)

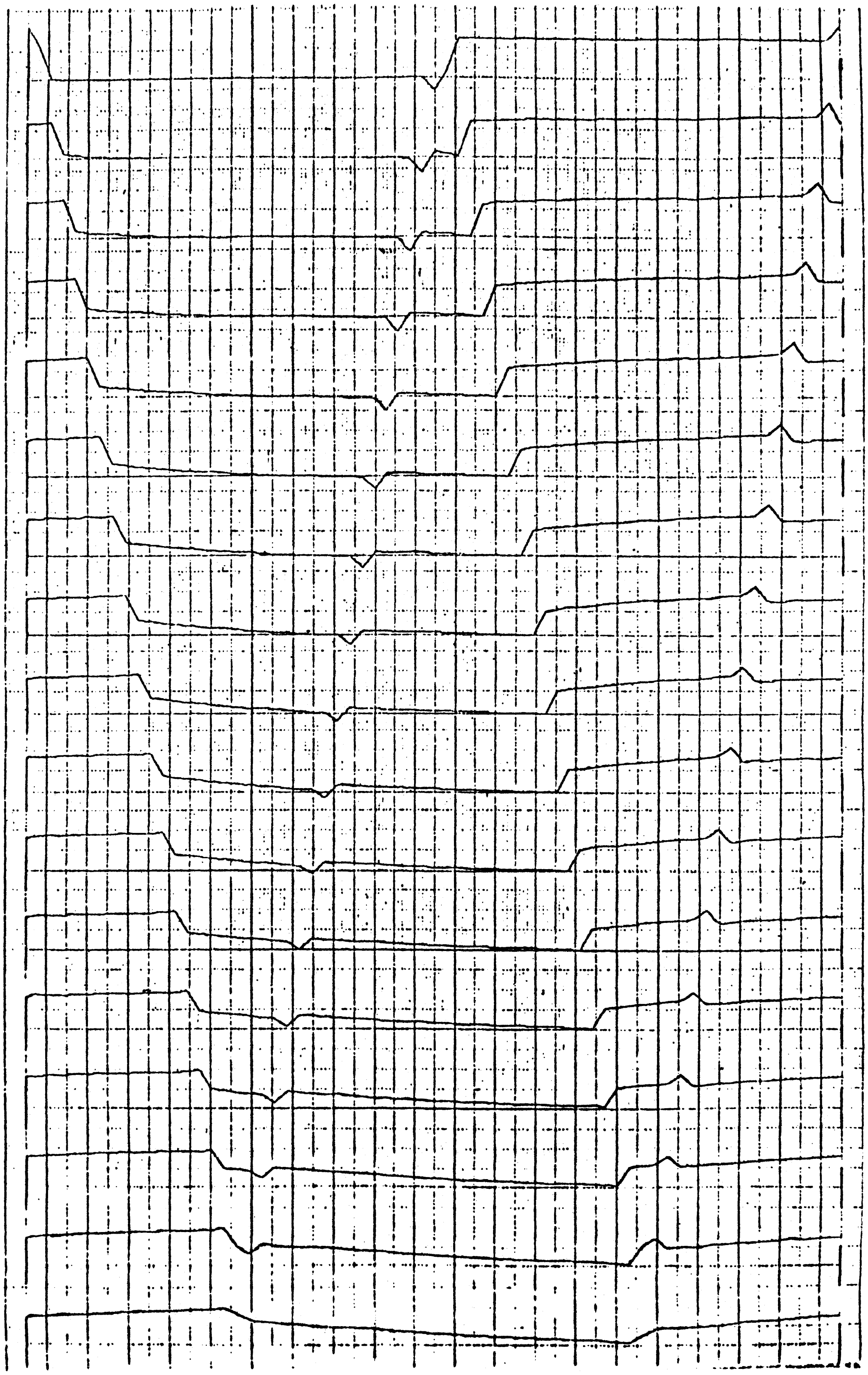

Fig.4 b)

Fig.4. Evolution of the weak shock wave coupled with heat flow. Governing equations are the same as those in Richtmyer and Morton's book[8], where $\mu/\nu=0.049381$ and $\nu=0.975933$. There is a double root, which is -0.953112 of index 2 at $k\Delta x=\pi$. No absolute eigenvalue is greater than 1.

a) The shock wave structure is completely hidden behind the error in the initial values from the second time step.

b) On the other hand, the initial values are smoothed. The shock wave structure can be observed in this figure.

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The author of the present paper suddenly passed away on December 27, 1975 by an accident.

A posteriori error estimation for Volterra integro-differential equations

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1. Introduction

In this paper we consider a posteriori error estimation of the approximate solution of the following nonlinear Volterra integro-differential equation

$$(1.1) \qquad \frac{dx}{dt} = f(t,x) + \int_0^t g(t,s,x(s))ds,$$

$$(1.2) \qquad x(0) = \eta, \quad 0 \leq t \leq T < +\infty.$$

Recently, for Volterra integro-differential equations as well as in the case of ordinary differential equations, various approximate methods, for example, the Chebyshev series method [1], the Linear multistep method [2], the Euler method [3], the Runge-Kutta method [4] and the Spline function method [5], have been reported. In them, though a priori error estimates to approximate solutions have been given, it seems that a posteriori error estimates have not been given to approximate solutions.

In the sequel, regarding equations (1.1)-(1.2) as an operator equation and applying the theorem on the Newton method for functional equations by Urabe [6], we prove the existence of an exact solution from an approximate solution and obtain a posteriori error estimate to the approximate solution. In this case this error bound is calculated by the uniform norm, and so larger than the exact one for each point. In the main Theorem 2 we obtain a pointwise error estimate under the weaker conditions than those in Theorem 1. To obtain the first approximate solution, we use the Picard iteration method discussed by Wolfe [1]. The merit of using the Chebyshev series is that its derivative or integral can be computed easily and the residual also can be easily. Though the Wolfe's method is essentially the application of the method of Clenshaw and Norton [7], the proper error estimate of the computational Chebyshev series to an exact solution was not discussed. On the other hand, in section 3 our main theorem guarantees the possibility of getting the proper error boun to an approximate solution obtained by Wolfe's method. In general, the speed of the convergence by the Picard iteration method is easier than the Newton method to get the computational solution. In our case, if we could not get the adjoint kernel X(t,s) explicitly, it is not easy to do the Newton iteration, but it is possible to do the theoretical Newton method and so get the better estimates than that of the Picard iteration method, provided that a starting approximate solution were obtained by the Chebyshev series iterated appropriately.

2. Preliminaries.

In this section we consider the nonlinear Volterra integro-differential system (1.1)-(1.2) under the following conditions:

(H)

$x(t)$: an unknown n-dimensional vector-valued function,

$f(t,x)$: a given n-dimensional continuous vector-valued function on I and continuously differentiable in $x \in D$,

$g(t,s,x)$: a given n-dimensional vector-valued function, which is continuous on (t,s) $(0 \leq s \leq t \leq T)$, continuously differentiable with respect to x in D and $g(t,s,x) = 0$ (if $t < s$), where D is a given domain of n-dimensional Euclidean space R^n and $I = [0,T]$.

Moreover we suppose

(2.1) $\phi(t,x) = \frac{\partial f(t,x)}{\partial x}$ and $\Phi(t,s,x) = \frac{\partial g(t,s,x)}{\partial x}$ are locally Lipschitz continuous in $x \in D$.

Let $C(I;D)$ be a space of all continuous bounded functions from I into D with the norm $\|x\| = \sup_{t \in I} |x(t)|$ for any $x \in C(I;D)$, where $|\cdot|$ denotes the Euclidean norm. Let $C^1(I;D)$ be a space of all continuously differentiable bounded functions from I into D. Furthermore we define a product space $V = C(I;R^n) \times R^n$. For any $y = [v(t), \xi] \in V$, if we define the norm $\|y\|_V = \|v\| + |\xi|$, then it can be easily shown that V is a Banach space.

Definition. For any $x(t) \in C^1(I;D)$, we define the residual operator $\rho(x)$:

(2.2) $$\rho(x)(t) = \frac{dx}{dt} - f(t,x(t)) - \int_0^t g(t,s,x(s))ds.$$

Under the above preparations, we consider the following operator

$$F(x)(t) = [\rho(x)(t), x(0) - \eta].$$

We also consider a linear operator from $C^1(I;D)$ into V such that

(2.3) $$L(x)(t) = [\frac{dx}{dt} - A(t)x(t) - \int_0^t a(t,s)x(s)ds,\ x(0)],$$

where $A(t)$ is an $n \times n$ continuous matrix on I and $a(t,s)$ an $n \times n$ continuous matrix on (t,s) $(0 \leq s \leq t \leq T)$, $a(t,s) = 0$ if $t < s$.

Lemma 1. The resolvent solution $X(t,s)$, which is the unique solution of the equation

$$\frac{\partial X(t,s)}{\partial s} = -X(t,s)A(s) - \int_s^t X(t,u)a(u,s)du,$$

$$X(t,t) = I, \qquad 0 \leq s \leq t \leq T,$$

is differentiable with respect to t and satisfies the equation

$$\frac{\partial X(t,s)}{\partial t} = A(t)X(t,s) + \int_s^t a(t,u)X(u,s)du, \qquad 0 \leq s \leq t \leq T.$$

And there exists an inverse operator L^{-1} for (2.3) such that

$$(2.4) \qquad L^{-1}(y)(t) = X(t,0)\xi + \int_0^t X(t,s)v(s)ds, \qquad y = [v, \xi].$$

Remark 1. It is well-known that if A(t) and a(t,s) are continuous functions, the adjoint kernel X(t,s) exists ([8]).

Thus under the suitable conditions we can establish the modified Newton method for the operator equation F(x) = 0 as follows:

$$(2.5) \qquad x_{n+1} = x_n - L^{-1}F(x_n) \qquad (n = 0,1,2,\ldots).$$

Applying the theorem on the Newton method for the functional equation by Urabe [6] to the system (1.1)-(1.2), we have a following existence theorem.

Theorem 1. Assume that the system (1.1)-(1.2) has an approximate solution $x = \bar{x}(t) \in D$, $\bar{x}(0) = \eta$, for which there are a positive number δ and a nonnegative number $\kappa \leq 1$ such that

(i) $D_\delta = \bigcup_{t\in I}\{x \mid |x - \bar{x}(t)| \leq \delta\} \subset D$,

(ii) $\sup_{t\in I}\{|\phi(t,x(t)) - A(t)| + \int_0^t |\Phi(t,s,x(s)) - a(t,s)|\, ds\} \leq \frac{\kappa}{M}$

for any $x \in C(I;D_\delta)$,

(iii) $\frac{Mr}{1 - \kappa} \leq \delta$,

where $r \geq 0$ and $M > 0$ are constants such that

$\|\rho(x)\| \leq r$ and $\|L^{-1}\| \leq M$.

Then the system (1.1)-(1.2) has one and only one solution $x = x(t) \in C^1(I;D_\delta)$ on I, and for this solution we have

$$\|\bar{x} - x\| \leq \frac{Mr}{1 - \kappa}$$

3. A pointwise error estimation

In Theorem 1 assumptions (2.1) and $0 \leq \kappa < 1$ are rather strong. In this section we obtain a theorem relating to the pointwise error estimation under the weaker assumptions than those in Theorem 1. In order to prove Theorem 2, we use the following lemma which could be easily proved.

Lemma 2. For a given nonnegative continuous function $u(t)$, $t \in I$, and a constant $c \geq 0$, we define a sequence $\{r_n(t;c)\}$ $(n = 0,1,2,\dots)$ as follows:

$$r_0(t;c) = \int_0^t u(s)ds, \qquad r_{n+1}(t,s) = c\int_0^t u(t-s)r_n(s;c)ds.$$

Then $\lim_{N\to\infty} \sum_{n=0}^{N} r_n(t;c) = R(t;c)$ exists uniformly on I, and $R(t;c)$ is nonnegative, nondecreaing, continuous and satisfies the following linear integral equation:

$$R(t;c) = r_0(t;c) + c\int_0^t u(t-s)R(s;c)ds, \qquad t \in I.$$

Furthermore if $c\int_0^t u(s)ds < 1$, then we have $R(t;c) \leq \dfrac{r_0(t;c)}{1 - c\int_0^t u(s)ds}$.

By using Lemmas 1 and 2, we get the following theorem.

Theorem 2. Under the hypothesis (H) we consider the Volterra integro-differential system (1.1)-(1.2). Suppose that the system (1.1)-(1.2) has an approximate solution $x = \bar{x}(t) \in D$, $\bar{x}(0) = \eta$, for which there are positive numbers δ and M, and a nonnegative number κ such that

(i) $\quad D_\delta = \bigcup_{t \in I} \{x \mid |x - \bar{x}(t)| \leq \delta\} \subset D,$

(ii) $\quad \sup_{t \in I} \{|\phi(t,x(t)) - A(t)| + \int_0^t |\Phi(t,s,x(s)) - a(t,s)|ds\} \leq \frac{\kappa}{M}$ for any $x \in C(I;D_\delta)$,

(iii) $\quad |X(t,s)| \leq u(t-s)$ and $\int_0^T u(s)ds \leq M,$

(iv) $\quad \|\rho(\bar{x})\| \leq r.$

Then if $rR(T;\frac{\kappa}{M}) \leq \delta$, the system (1.1)-(1.2) has one and only one solution $x = x(t) \in C^1(I;D_\delta)$, and for this solution we have

$$|\bar{x}(t) - x(t)| \leq rR(t;\frac{\kappa}{M}), \qquad t \in I,$$

and

$$|\rho(x_n)(t)| \leq r\frac{\kappa}{M} r_{n-1}(t;\frac{\kappa}{M}) \qquad \text{if } \kappa \neq 0 \ (n = 0,1,2,\ldots),$$
$$\rho(x_n)(t) = 0 \qquad \text{if } \kappa = 0 \ (n = 1,2,\ldots),$$

where $r_{-1}(t;\frac{\kappa}{M}) = \frac{M}{\kappa}$ if $\kappa \neq 0$ and $\{x_n\}$ $(n = 0,1,2,\ldots)$ is defined in (2.5) with $x_0 = \bar{x}$.

Proof. At first from the definition of $\{x_n\}$ we have

$$(3.1) \qquad x_{n+1}(t) - x_n(t) = -\int_0^t X(t,s)\rho(x_n)(s)ds \quad (n = 0,1,\ldots).$$

The proof for $\varkappa = 0$ is essentially the same in the case of $\varkappa \neq 0$. So we prove the theorem only for $\varkappa \neq 0$. Suppose that $\varkappa \neq 0$. Then we obtain by mathematical induction

(3.2) $\qquad |\rho(x_n)(t)| \leq r\frac{\varkappa}{M}\, r_{n-1}(t;\frac{\varkappa}{M})$

and

(3.3) $\qquad |x_{n+1}(t) - x_n(t)| \leq rr_n(t;\frac{\varkappa}{M})$ for $n = 0,1,2,\ldots$.

In fact the results are obvious for $n = 0$. Assume that $x_1, x_2, \ldots, x_N$ are all in D_δ and (3.2), (3.3) hold for $n = 0,1,2,\ldots,N$. From (3.3) it follows that $|x_{N+1}(t) - x_0(t)| \leq \delta$, that is, $x_{N+1}(t) \in D_\delta$, $t \in I$. Then the definition of $\rho(x)$, Lemma 1, (3.1), the induction hypothsis and the mean value theorem imply that

$$|\rho(x_{N+1})(t)| = \left|\frac{dx_{N+1}}{dt} - f(t,x_{N+1}(t)) - \int_0^t g(t,s,x_{N+1}(s))ds\right|$$

$$= \left|\frac{dx_N}{dt} - \rho(x_N)(t) - \int_0^t \frac{\partial X(t,s)}{\partial t}\rho(x_N)(s)ds - f(t,x_{N+1}(t))\right.$$

$$\left. - \int_0^t g(t,s,x_{N+1}(s))ds\right|$$

$$= |(f(t,x_N(t)) - f(t,x_{N+1}(t)) + A(t)(x_{N+1}(t) - x_N(t))$$

$$+ \int_0^t \{g(t,s,x_N(s)) - g(t,s,x_{N+1}(s)) + a(t,s)(x_{N+1}(s) - x_N(s))\}ds|$$

$$\leq \int_0^1 \{|A(t) - \emptyset(t,x_N(t) + \theta(x_{N+1}(t) - x_N(t)))| + \int_0^t |a(t,s)$$

$$- \Phi(t,s,x_N(s) + \theta(x_{N+1}(s) - x_N(s)))|ds\}d\theta\; rr_N(t;\frac{\varkappa}{M})$$

$$\leq r\frac{\kappa}{M} r_N(t;\frac{\kappa}{M}),$$

and

$$|x_{N+2}(t) - x_{N+1}(t)| \leq |\int_0^t X(t,s)\rho(x_{N+1})(s)ds|$$

$$\leq r\frac{\kappa}{M}\int_0^t u(t-s)r_N(s;\frac{\kappa}{M})ds$$

$$= rr_{N+1}(t;\frac{\kappa}{M}).$$

Therefore (3.2) and (3.3) are proved for all n. As the remainders of the proof are similar to those in Theorem 1 in [6], they are omitted here.

Corollary 2.1. Under the same assumptions as in Theorem 2, we have

$$|x_n(t) - x(t)| \leq \|\rho(x_n)\| R(t;\frac{\kappa}{M})$$

$$\leq r\frac{\kappa}{M}\|r_{n-1}(\cdot,\frac{\kappa}{M})\| R(t;\frac{\kappa}{M})$$

for $\kappa \neq 0$ and $n = 0,1,2,\dots$.

The proof of Corollary 2.1 follows immediately from the proof of Theorem 2 if we regard $\rho(x_n)$ as r in Theorem 2.

Corollary 2.2. In Theorem 2, if κ satisfies the inequality $\kappa<1$, we have

$$|rR(t;\frac{\kappa}{M})| \leq \frac{Mr}{1-\kappa}, \qquad \|\bar{x} - x\| \leq \frac{Mr}{1-\kappa}.$$

This is just the result of Theorem 1. The proof of Corollary 2.2

follows from Lemma 2 and Theorem 2.

Corollary 2.3. Under the same assumptions as in Corollary 2.2, we have

$$|\rho(x_n)| \leq r\kappa^n, \quad |x_n(t) - x(t)| \leq r\kappa^n R(t;\frac{\kappa}{M})$$

for n = 0,1,2,... .

The proof of Corollary 2.3 follows easily from our hypothesis and the definition of r_n.

Corollary 2.4. If the adjoint kernel X(t,s) satisfies the inequality

$$|X(t,s)| \leq c \exp(b(t-s)), \quad t \in I,$$

where b and c are some constants, then we obtain

$$R(t;\frac{\kappa}{M}) = \begin{cases} \dfrac{c}{b + \frac{\kappa}{M}c} (\exp((b+\frac{\kappa}{M}c)t) - 1) & \text{if } b+\frac{\kappa}{M}c \neq 0, \\ ct & \text{if } b+\frac{\kappa}{M}c = 0. \end{cases}$$

If we solve the integral equation

$$R(t) = (c\int_0^t \exp(bs)ds) + \frac{c\kappa}{M} \int_0^t \exp(b(t-s))R(s)ds ,$$

then we find the result of Corollary 2.4.

4. Computational procedure and numerical examples

In his paper [1], Wolfe established the Picard iteration method with a Chebyshev interpolation series to compute numerically the solutions of (1.1)-(1.2). In this paper we used his procedure to provide a first approximate solution in the following examples. Though, in general, it is difficult to find the residual for $\bar{x}$ exactly, it is easy to compute it approximately, provided that a first approximate solution $\bar{x}$ were obtained by Wolfe's procedure. In fact, from the definition of the residual function we find $\rho(\bar{x})(t)$ so that neglecting truncation errors,

$$\rho(\bar{x})(t) = \frac{dy_N}{dt} - \frac{dy_{N+1}}{dt}$$

$$= {\sum_{n=0}^{N}}' (A_{N,j} - A_{N+1,j})T_j(t) + A_{N+1,N+1}T_{N+1}(t)$$

where $\frac{dy_N}{dt} = {\sum_{n=0}^{N}}' A_{N,j}T_j(t)$ is the N-times computationally iterated function by means of Wolfe's procedure and $\bar{x} = y_N$.

Wolfe's procedure is easily extended to solve the adjoint kernel $X(t,s)$ numerically with a double Chebyshev representation. Thus the Newton iteration (3.1) could be carried out computationally. This procedure has not been executed in this paper because it is primarily our purpose to obtain an a posteriori error estimate for Volterra integro-differential equations.

We now give two examples. Especially the first has been used by many authors [9].

Example 1.

$$\frac{dx}{dt} = 1 + 2t - x(t) + \int_0^t t(1+2t)e^{s(t-s)}x(s)ds,$$

$$x(0) = 1, \quad 0 \leq t \leq 1.$$

Chebyshev coefficients of $\bar{x}$

a_0	3.118449253672	a_8	0.000006752824
a_1	0.800895417852	a_9	0.000000846036
a_2	0.287218928325	a_{10}	0.000000105245
a_3	0.056192726040	a_{11}	0.000000011974
a_4	0.012351784489	a_{12}	0.000000001343
a_5	0.002004280830	a_{13}	0.000000000141
a_6	0.000338715218	a_{14}	0.000000000015
a_7	0.000047631421	a_{15}	0.000000000001
		a_{16}	0.000000000000

Starting order of the Chebyshev series : 4 .

The number of the Picard iteration : 12 .

(1) Error estimates by Theorem 1:

$M = 178$, $r = 5.2 \times 10^{-11}$, $\kappa = 0$, $\|\bar{x} - \hat{x}\| \leq 1.068 \times 10^{-8}$.

(2) Error estimates by Theorem 2:

$|\bar{x}(t) - \hat{x}(t)| \leq 0.2r(e^{5t} - 1), \quad 0 \leq t \leq 1,$

$x_n(t) = \hat{x}(t) = \exp(t^2) \quad (n = 1,2,\ldots).$

cf. $\bar{x}(1) = 2.718281828590$,

$\hat{x}(1) = 2.718281828459$,

(Theorem 2) $|\bar{x}(1) - \hat{x}(1)| \leq 1.540 \times 10^{-9}$.

Example 2.

$$\frac{dx}{dt} = x(t) - 0.5 \int_0^t e^{-(t-s)} x^2(s)ds,$$

$$x(0) = 0.02, \quad 0 \leq t \leq 2.$$

Chebyshev coefficients of $\bar{x}$

a_0	0.1360872	a_4	0.0002507
a_1	0.0602175	a_5	0.0000205
a_2	0.0141604	a_6	0.0000010
a_3	0.0022177	a_i	0.0 $(7 \leq i \leq 29)$

Starting order of the Chebyshev series : 10 .

The number of iterations : 19 .

(1) Error estimates by Theorem 1:

$M = 15.5731$, $r = 1.660 \times 10^{-6}$, $\kappa = 3.483 \times 10^{-4}$,

$\|\bar{x} - \hat{x}\| \leq 2.5861 \times 10^{-5}$.

(2) Error estimates by Theorem 2:

$M = 7.032$, $r = 1.660 \times 10^{-6}$, $\kappa = 7.098 \times 10^{-5}$,

$|\bar{x}(t) - \hat{x}(t)| \leq 1.55 \times 10^{-6}(e^{1.0725t} - 1)$, $0 \leq t \leq 2$.

cf. $|\bar{x}(2) - \hat{x}(2)| \leq 1.17 \times 10^{-5}$.

All the computations were carried out on the digital computer IBM 370 MODEL/135 at the Computation Center, Waseda University.

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On the Existence of an Approximate Solution in Chebyshev Series of a Nonlinear Integral Equation of Fredholm Type

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Abstract

In the present paper we study a nonlinear integral equation of Fredholm type and prove the existence and the uniqueness of an approximate solution in the form of finite Chebyshev series accurately as it is desired for an isolated solution of the integral equation. We use Galerkin's procedure based on Chebyshev polynomials and determine the coefficients of the series by the method of Newton-Raphson's to obtain the desired approximate solution. The proof we give is analoguous to that by M. Urabe [3].

0. Introduction

M. Urabe [3] studied multi-point boundary value problems for nonlinear ordinary differential equations and proved three basic theorems (Theorem 1, Theorem 2 and Theorem 3) on approximate solutions in Chebyshev series of the boundary value problems. He used Galerkin's procedure based on Chebyshev polynomials and determined the coefficients of the finite Chebyshev series by the method of Newton-Raphson's to obtain the approximate solution.

Theorem 1 says that for any isolated solution there exists an approximate solution accurately as it is desired by computing finite Chebyshev polynomial series. Theorem 2 says that the obtained Chebyshev approximation corresponds one to one to the isolated solution. Theorem 3, which plays an important role in practical applications, says that one can always assure the existence of an exact isolated solution by checking several conditions on the obtained Chebyshev approximation and further it gives a method to obtain an error bound of the obtained approximate solution.

It is expected that the analoguous conclusion will be obtained for nonlinear integral equations of Fredholm type. In fact, Shimasaki M. and T. Kiyono [1] obtained the numerical solutions of the nonlinear integral equations using the method of Chebyshev series analoguous to that by M. Urabe [3]. They gave some numerical examples and stated a fundamental theorem analoguous to Theorem 3 to obtain the error bounds of these numerical solutions.

In the present paper we prove two theorems analoguous to Theorem 1 and Theorem 2 by M. Urabe [3] on the existence and the uniqueness of an approximate solution in the form of finite Chebyshev polynomial series for an isolated solution. We consider a nonlinear integral equation of Fredholm type of the form:

$$(0.\ 1) \qquad u(t)=f(t)+\int_{-1}^{1} K(t,s,u(s))ds$$

on the interval [-1,1]. Denote by J the closed interval [-1,1].

Let D be an open interval. Here we assume on the equation (0. 1) the conditions that f(t) is continuously differentiable function of t on the interval J and that K(t,s,u) is continuous function of t, s and u on the region $J\times J\times\bar{D}$ and also twice continuously differentiable function of the arguments t and u on the same region.

In order to obtain an approximate solution of the equation (0. 1), we take the finite Chebyshev polynomial series with unknown coefficients a_n $(n=0,1,\cdots\cdots,m)$ such that

$$(0.\ 2)\qquad u_m(t)=\sum_{n=0}^{m} e_n a_n T_n(t).$$

Here we denote by $T_n(t)$ the Chebyshev polynomial of degree n defined in the following form:

$$(0.\ 3)\qquad T_n(t)=\cos(n\cos^{-1}t)\qquad \text{for } t\in J \text{ and } n=0,1,$$

and also denote by e_n the constant number of the form:

$$(0.\ 4)\qquad e_0=1,\ e_n=\sqrt{2}\qquad \text{for } n=1,2,\cdots\cdots.$$

It will be reasonable to determine these m+1 coefficients a_n $(n=0,1,\cdots\cdots,m)$ so that

$$(0.\ 5)\qquad u_m(t)=(P_m f)(t)+P_m\int_{-1}^{1}K(t,s,u_m(s))ds$$

may hold. Here P_m is the operator which expresses the truncation of the Chebyshev series of the operand discarding the terms of the order higher than m. In what follows, the finite Chebyshev series $u_m(t)$ satisfying the equation (0. 5) will be called Chebyshev approximation of order m.

According to the definition by Shimasaki M. and T. Kiyono [1], we call $u=\hat{u}(t)$ isolated solution of the equation (0. 1) when it is a solution of the equation (0. 1) the first variational equation of which

$$(0.\ 6) \qquad v(t)-\int_{-1}^{1}\frac{\partial K}{\partial u}(t,s,\hat{u}(s))v(s)ds=0$$

has no continuous solution except $v(t)\equiv 0$. The nomenclature comes from the fact that for any isolated solution $u=\hat{u}(t)$ of the equation (0. 1) there is no other solution of the equation (0. 1) in a sufficiently small neighborhood of $u=\hat{u}(t)$. (see [1]).

The conclusion of the present paper is the following two theorems.

Theorem 1. If the equation (0. 1) has an isolated solution $u=\hat{u}(t)$ lying in an open interval D for any $t\in J$, then for sufficiently large m_0 there exists a Chebyshev approximation $u=\bar{u}_m(t)$ of any order $m\geqq m_0$ such that the sequence $\bar{u}_m(t)$ converges uniformly to the solution $\hat{u}(t)$ on the interval J.

Theorem 2. The Chebyshev approximation $u=\bar{u}_m(t)$ stated in Theorem 1 is determined uniquely in a sufficiently small neighborhood of $u=\hat{u}(t)$ provided the order m of the Chebyshev approximation $u=\bar{u}_m(t)$ is sufficiently high.

In order to prove Theorem 1, we use the following lemma proved by M. Urabe based on the Newton-Raphson's procedure to determine the coefficients of desired Chebyshev approximation.

Lemma. (M.Urabe [2] and [3]) Let

(0. 7) $F(\alpha)=0$

be a given real system of equations, where α and $F(\alpha)$ are vectors of the same dimension and $F(\alpha)$ is a continuously differentiable function of α defined in some region Ω of the α-space. Assume that (0. 7) has an approximate solution $\alpha=\hat{\alpha}$ for which the determinant of the Jacobian matrix $J(\alpha)$ of $F(\alpha)$ with respect to α does not vanish and that there are positive constant δ and a nonnegative constant $\kappa<1$ such that

(1) $\Omega_\delta=\{\alpha \mid \|\alpha-\hat{\alpha}\| \leqq \delta\} \subset \Omega$

(2) $\|J(\alpha)-J(\hat{\alpha})\| \leqq \kappa/M'$ for any $\alpha \in \Omega_\delta$

(3) $M'r/(1-\kappa) \leqq \delta$,

where r and M' are numbers such that

$$\|F(\hat{\alpha})\| \leqq r \quad \text{and} \quad \|J^{-1}(\hat{\alpha})\| \leqq M'.$$

Then the system (0. 7) has one and only one solution $\alpha=\bar{\alpha}$ in Ω_δ and for $\alpha=\bar{\alpha}$ it holds that

$$\det J(\bar{\alpha}) \neq 0 \quad \text{and} \quad \|\bar{\alpha}-\hat{\alpha}\| \leqq M'r/(1-\kappa).$$

Here we denote by the symbol $\| \ \|$ Euclidean norms for vectors and matrices.

1. Some Properties of Chebyshev Series

Denote by $C(J)$ the family of all continuous functions of t on the interval J. For any function $f=f(t)\in C(J)$ we use the two kinds of norms $\|f\|_n$ and $\|f\|_q$, which are defined as follows:

$$(1.\ 1)\qquad \|f\|_n=\sup_{t\in J}|f(t)|$$

and

$$(1.\ 2)\qquad \|f\|_q=\left[\frac{1}{\pi}\int_{-1}^{1}|f(t)|^2(1-t^2)^{-1/2}dt\right]^{1/2}.$$

When we use the notations (0. 3) and (0. 4), it is known that any function $f=f(t)\in C(J)$ is expanded in Chebyshev series of the form:

$$(1.\ 3)\qquad f(t)\sim\sum_{n=0}^{\infty}e_n a_n T_n(t),$$

where

$$a_n=\frac{1}{\pi}e_n\int_{-1}^{1}f(t)T_n(t)(1-t^2)^{-1/2}dt.$$

Applying Parseval's equality to the expansion (1. 3) and using the definition (1. 2), we have

$$(1.\ 4)\qquad \|f\|_q^2=\sum_{n=0}^{\infty}|a_n|^2.$$

In particular for finite Chebyshev series of the form:

$$f_m(t)=\sum_{n=0}^{m}e_n a_n T_n(t),$$

the equality (1. 4) implies that

$$(1.\ 5)\qquad \|f_m\|_q=\|\alpha\|,$$

where α is the vector $\alpha=(a_0,a_1,\cdots\cdots,a_m)$.

Moreover using the equality (1. 5) and Schwarz's inequality for the same finite Chebyshev series, we have

(1. 6) $\|f_m\|_n \leq \sqrt{2m+1}\|\alpha\|$.

By the definition of the operator $P_m: C(J)\to C(J)$ for any $f=f(t)\in C(J)$ we have

$$(P_m f)(t)=\sum_{n=0}^{m} e_n a_n T_n(t).$$

Consequently it follows that

$$((I-P_m)f)(t)\sim \sum_{n=m+1}^{\infty} e_n a_n T_n(t),$$

where I denotes the identity operator. If $f=f(t)$ is continuously differentiable function of t on the interval J, it is proved that

(1. 7) $\|(I-P_m)f\|_n \leqq \sigma(m)\|(I-P_{m-1})\dot{f}\|_q \leqq \sigma(m)\|\dot{f}\|_q$

and

(1. 8) $\|(I-P_m)f\|_q \leqq \sigma_1(m)\|(I-P_{m-1})\dot{f}\|_q \leqq \sigma_1(m)\|\dot{f}\|_q$

for $m=0,1,\cdots\cdots$, where $P_{-1}=0$, $\dot{f}=df/dt$ and the functions $\sigma(m)$ and $\sigma_1(m)$ are monotone deceasing of m satisfying respectively

(1. 9) $\frac{\sqrt{2}}{m+1} \leqq \sigma(m) \leqq \sqrt{2}/\sqrt{m}$ and $\sigma_1(m)=\frac{1}{m+1}$.

These properties of Chebyshev series was proved in detail in the paper by M. Urabe [3].

2. Fundamental Inequalities

If $u=\hat{u}(t)$ is an isolated solution of the equation (0. 1) lying in the interval D for any $t\in J$, there exists a positive number δ such that

$$D_\delta=\{u|\,|u-\hat{u}(t)|\leqq\delta \text{ for some } t\in J\}\subset D.$$

Denote that $\hat{u}_m=P_m\hat{u}$. It follows from the inequalities (1. 7) and (1. 8) that

$$(2.\ 1)\qquad \|\hat{u}_m-\hat{u}\|_n\leqq\sigma(m)\left\|\frac{d\hat{u}}{dt}\right\|_q\leqq M_1\sigma(m)$$

and

$$(2.\ 2)\qquad \|\hat{u}_m-\hat{u}\|_q\leqq\sigma_1(m)\left\|\frac{d\hat{u}}{dt}\right\|_q\leqq M_1\sigma_1(m),$$

where

$$M_1=\max_J\left|\frac{df}{dt}(t)\right|+2\max_{J\times J\times\bar{D}}\left|\frac{\partial K}{\partial t}(t,s,u)\right|.$$

Hence from the inequalities (1. 9) and (2. 1) there exists a number m_1 sufficiently large such that for any $m\geqq m_m$

$$\hat{u}_m(t)\in D_\delta\subset D \qquad \text{for any } t\in J.$$

The coefficients $\alpha=(a_0,a_1,\cdots\cdots,a_m)$ of our desired Chebyshev approximation of order m

$$u_m(t)=\sum_{n=0}^{m} e_n a_n T_n(t)$$

will be determined from the equation (0. 5), that is equivalent to the system of nonlinear algebraic equations

$$(2.\ 3)\qquad F^{(m)}(\alpha)=(F_0(\alpha),F_1(\alpha),\cdots\cdots,F_m(\alpha))=0,$$

where

$$(2.\ 4)\qquad u_m(t)-(P_mf)(t)-P_m\int_{-1}^{1}K(t,s,u_m(s))ds=\sum_{n=0}^{m}e_nF_n(\alpha)T_n(t).$$

The equation (2. 3) is called determining equation of Chebyshev approximation.

In order to determine a domain where the function $F^{(m)}(\alpha)$ is well defined, we take a number $m_2 \geqq m_1$ such that for any $m \geqq m_2$

$$V_m = \{u \mid |u - \hat{u}_m(t)| \leqq \delta - M_1\sigma(m) \text{ for some } t \in J\} \subset D_\delta \subset D$$

since

$$|u - \hat{u}(t)| \leqq |u - \hat{u}_m(t)| + |\hat{u}_m(t) - \hat{u}(t)| \leqq \delta - M_1\sigma(m) + \|\hat{u}_m - \hat{u}\|_n$$

for any $u \in V_m$. Let us put that

$$\hat{u}(t) = \sum_{n=0}^{\infty} e_n a_n T_n(t) \qquad \text{and} \qquad \hat{\alpha} = (\hat{a}_0, \hat{a}_1, \cdots\cdots, \hat{a}_m)$$

and define the domain of the form

$$\Omega_m = \{\alpha \mid \|\alpha - \hat{\alpha}\| \leqq \frac{1}{\sqrt{2m+1}}(\delta - M_1\sigma(m))\}.$$

For any $\alpha = (a_0, a_1, \cdots\cdots, a_m) \in \Omega_m$,
we obtain

$$u_m(t) = \sum_{n=0}^{m} e_n a_n T_n(t) \in V_m \subset D \qquad \text{for any } t \in J$$

since

$$\|u_m - \hat{u}_m\|_n \leqq \sqrt{2m+1}\|\alpha - \hat{\alpha}\| \leqq \delta - M_1\sigma(m)$$

from the inequality (1. 6). Therefore it is concluded that the function $F^{(m)}(\alpha)$ is defined on the domain Ω_m and continuously differentiable function of α on the same domain from the definition (2. 4).

Let $J_m(\alpha)$ be the Jacobian matrix of the function $F^{(m)}(\alpha)$. To investigate the properties of the matrix $J_m(\alpha)$, let us consider a linear equation of the form

$$(2.\ 5)\qquad J_m(\alpha)\xi=\eta,$$

where

$$\alpha=(a_0,a_1,\cdots\cdots,a_m)\in\Omega_m,$$

$$\xi=(x_0,x_1,\cdots\cdots,x_m)\quad\text{and}\quad\eta=(y_0,y_1,\cdots\cdots,y_m).$$

If we put

$$u_m(t)=\sum_{n=0}^{m}e_n a_n T_n(t),$$

$$v(t)=\sum_{n=0}^{m}e_n x_n T_n(t)\quad\text{and}\quad w(t)=\sum_{n=0}^{m}e_n y_n T_n(t),$$

then by the definition of the Jacobian matrix $J_m(\alpha)$ we have

$$(2.\ 6)\qquad v(t)-P_m\int_{-1}^{1}\frac{\partial K}{\partial u}(t,s,u_m(s))v(s)ds=w(t)$$

corresponding to the linear equation (2. 5).

Substituting $\hat{u}_m(t)$ for $u_m(t)$, we rewrite the equation (2. 6) in the form

$$(2.\ 7)\qquad v(t)-\int_{-1}^{1}\frac{\partial K}{\partial u}(t,s,\hat{u}(s))v(s)ds=w(t)+R(t),$$

where

$$R(t)=-(I-P_m)\int_{-1}^{1}\frac{\partial K}{\partial u}(t,s,\hat{u}(s))v(s)ds$$
$$-P_m\int_{-1}^{1}[\frac{\partial K}{\partial u}(t,s,\hat{u}(s))-\frac{\partial K}{\partial u}(t,s,\hat{u}_m(s))]v(s)ds.$$

It is easy to prove from the inequality (1. 8) and the Parseval's equality (1. 4) that

$$(2.\ 8)\qquad \|R\|_q\leqq\sigma_1(m)\|\frac{d}{dt}\int_{-1}^{1}\frac{\partial K}{\partial u}(t,s,\hat{u}(s))v(s)ds\|_q$$

$$+\left\|\int_{-1}^{1}\left[\frac{\partial K}{\partial u}(t,s,\hat{u}(s))-\frac{\partial K}{\partial u}(t,s,\hat{u}_m(s))\right]v(s)ds\right\|_q$$

$$\leqq M_2\sigma_1(m)\|v\|_q+M_3\|\hat{u}-\hat{u}_m\|_q\|v\|_q\leqq(M_2+M_3M_1)\sigma_1(m)\|v\|_q,$$

where

$$M_2=\sqrt{2\pi}\max_{J\times J\times\bar{D}}\left|\frac{\partial^2 K}{\partial u\partial t}(t,s,u)\right|$$

and

$$M_3=\pi\max_{J\times J\times\bar{D}}\left|\frac{\partial^2 K}{\partial u^2}(t,s,u)\right|.$$

From the definition of the isolated solution $u=\hat{u}(t)$ and well-known theory for linear integral equations of Fredholm type it follows that there exists a constant number M such that

$$\|v\|_q\leqq M\|w+R\|_q$$

for the equation (2. 7). (see [1] or [4]) Therefore from the inequality (2. 8) we have

$$\|v\|_q\leqq M(\|w\|_q+\|R\|_q)\leqq M\|w\|_q+M(M_2+M_3M_1)\sigma_1(m)\|v\|_q.$$

If we take $m_3\geqq m_2$ sufficiently large, we obtain for any $m\geqq m_3$

$$\|v\|_q\leqq\frac{M}{1-M(M_2+M_3M_1)\sigma_1(m)}\|w\|_q.$$

By the equality (1. 5) this is equivalent to the inequality

$$\|\xi\|\leqq\frac{M}{1-M(M_2+M_3M_1)\sigma_1(m)}\|\eta\|. \tag{2. 9}$$

It readily follows from the equation (2. 5) that for any $m\geqq m_3$

$$\det J_m(\hat{\alpha})\neq 0 \tag{2.10}$$

and

$$\|J_m^{-1}(\hat{\alpha})\|\leqq\frac{M}{1-M(M_2+M_3M_1)\sigma_1(m)}\leqq M', \tag{2.11}$$

where M' is a constant number. In fact, put $\eta=0$ in (2. 5), then by (2. 9) we have $\xi=0$, which implies (2.10). By (2.10), from (2. 5) we have

$$\xi=J_m^{-1}(\hat{\alpha})\eta.$$

Then by (2. 9) we have (2.11). This inequality (2.11) plays an important role in the proof of Theorem 1.

Let

$$\alpha'=(a_0',a_1',\cdots\cdots,a_m') \quad \text{and} \quad \alpha''=(a_0'',a_1'',\cdots\cdots,a_m'')$$

be arbitrary vectors belonging to the domain Ω_m. Then both

$$u_m'(t)=\sum_{n=0}^{m} e_n a_n' T_n(t) \quad \text{and} \quad u_m''(t)=\sum_{n=0}^{m} e_n a_n'' T_n(t)$$

lie in $V_m \subset D$ for any $t \in J$. For any vector $\xi=(x_0,x_1,\cdots\cdots,x_m)$ let us put

$$J_m(\alpha')\xi=\eta' \quad \text{and} \quad J_m(\alpha'')\xi=\eta'', \tag{2.12}$$

where

$$\eta'=(y_0',y_1',\cdots\cdots,y_m') \quad \text{and} \quad \eta''=(y_0'',y_1'',\cdots\cdots,y_m'').$$

Corresponding to the equations (2.12), we have

$$v(t)-P_m\int_{-1}^{1}\frac{\partial K}{\partial u}(t,s,u_m'(s))v(s)ds=w'(t) \tag{2.13}$$

and

$$v(t)-P_m\int_{-1}^{1}\frac{\partial K}{\partial u}(t,s,u_m''(s))v(s)ds=w''(t),$$

where

$$v(t)=\sum_{n=0}^{m} e_n x_n T_n(t),$$

$$w'(t)=\sum_{n=0}^{m} e_n y_n' T_n(t) \qquad \text{and} \qquad w''(t)=\sum_{n=0}^{m} e_n y_n'' T_n(t).$$

By the equation (2.13) and (2.14) we have

$$w'(t)-w''(t)=-P_m\int_{-1}^{1}[\frac{\partial K}{\partial u}(t,s,u_m'(s))-\frac{\partial K}{\partial u}(t,s,u_m''(s))]v(s)ds.$$

Then it is easy to prove from the Parseval's equality (1. 4) and Schwarz's inequality that

$$\|w'-w''\|_q \leqq \|\int_{-1}^{1}[\frac{\partial K}{\partial u}(t,s,u_m'(s))-\frac{\partial K}{\partial u}(t,s,u_m''(s))]v(s)ds\|_q$$

$$\leqq M_3\|u_m'-u_m''\|_q\|v\|_q.$$

On the other hand, it follows from the equality (1. 5) and the equations (2.12) that

$$\|u_m'-u_m''\|_q=\|\alpha'-\alpha''\|, \qquad \|v\|_q=\|\xi\|$$

and

$$\|w'-w''\|_q=\|\eta'-\eta''\|=\|J_m(\alpha')\xi-J_m(\alpha'')\xi\|.$$

Hence we have

$$\|[J_m(\alpha')-J_m(\alpha'')]\xi\|\leqq M_3\|\alpha'-\alpha''\|\|\xi\|,$$

which implies

(2.15) $\qquad \|J_m(\alpha')-J_m(\alpha'')\|\leqq M_3\|\alpha'-\alpha''\|$

for all vectors α' and α'' belonging to the domain Ω_m. This inequality (2.15) also plays an important role in the proof of Theorem 1.

3. Proof of Theorem 1

Let $u=\hat{u}(t)$ be an isolated solution of the equation (0. 1) lying in the interval D for any $t\in J$. Denote that $\hat{u}_m=P_m\hat{u}$ and put

$$\hat{u}_m(t)=\sum_{n=0}^{m} e_n\hat{a}_nT_n(t) \qquad \text{and} \qquad \hat{\alpha}=(\hat{a}_0,\hat{a}_1,\cdots\cdots,\hat{a}_m).$$

It is concluded from the previous section that there exists a number m_3 such that for any $m\geqq m_3$ the function $F^{(m)}(\alpha)$ defined in (2. 3) and (2. 4) is continuously differentiable of α on the domain Ω_m and the Jacobian matrix $J_m(\alpha)$ of the function $F^{(m)}(\alpha)$ has the inverse $J_m^{-1}(\alpha)$ at $\alpha=\hat{\alpha}$ satisfying the inequality (2.11) and satisfies (2.15).

Now we will apply the Lemma in the section 0 to the equation (2. 3), the roots of which are the coefficients of our desired Chebyshev approximation. For any $m\geqq m_3$ let us put

$$\hat{u}_m(t)-(P_mf)(t)-P_m\int_{-1}^{1}K(t,s,\hat{u}_m(s))ds=R_m(t).$$

This is rewritten in the form

$$R_m(t)=P_m\int_{-1}^{1}[K(t,s,\hat{u}(s))-K(t,s,\hat{u}_m(s))]ds$$

and hence from the Parseval's equality (1. 4) and Schwarz's inequality it follows that

$$\|R_m\|_q\leqq\left\|\int_{-1}^{1}[K(t,s,\hat{u}(s))-K(t,s,\hat{u}_m(s))]ds\right\|_q$$
$$\leqq M_4\|\hat{u}_m-\hat{u}\|_q\leqq M_4M_1\sigma_1(m),$$

where

$$M_4=\sqrt{2\pi}\max_{J\times J\times\bar{D}}\left|\frac{\partial K}{\partial u}(t,s,u)\right|.$$

If we put

$$F^{(m)}(\hat{\alpha})=\rho,$$

then we have from the equality (1. 5)

(3. 1) $$\|\rho\|=\|F^{(m)}(\hat{\alpha})\|=\|R_m\|_q\leq M_1M_4\sigma_1(m).$$

Since $\sigma_1(m)=(m+1)^{-1}$, the inequality (3. 1) expresses that $\alpha=\hat{\alpha}$ is an approximate solution of the determining equation (2. 3) for any sufficiently large m.

In order to check the conditions (1), (2) and (3) in Lemma in the section 0, we take an arbitrary nonnegative number $\kappa<1$ and put

$$\delta_1=\min\{\frac{\kappa}{M_3M'},\ \delta-M_1\sigma(m_3)\}.$$

There exists a number $m_4\geqq m_3$ so that

$$[M'M_1M_4/(1-\kappa)]\sigma_1(m)<\delta_1/\sqrt{2m+1}$$

may hold for any $m\geqq m_4$ since $\sqrt{2m+1}\sigma_1(m)=O(m^{-1/2})$ as $m\to\infty$. If we take δ_m such that

(3. 2) $$[M'M_1M_4/(1-\kappa)]\sigma_1(m)<\delta_m<\delta_1/\sqrt{2m+1},$$

(3. 3) $$\Omega_{\delta_m}=\{\alpha\,|\,\|\alpha-\hat{\alpha}\|\leqq\delta_m\}\subset\Omega_m.$$

In fact, for any $\alpha\in\Omega_{\delta_m}$ and any $m\geqq m_4$

$$\|\alpha-\hat{\alpha}\|\leqq\delta_m\leqq\delta_1/\sqrt{2m+1}$$

$$\leqq[\delta-M_1\sigma(m_3)]/\sqrt{2m+1}\leqq[\delta-M_1\sigma(m)]/\sqrt{2m+1},$$

which implies $\alpha\in\Omega_m$. Moreover for any $\alpha\in\Omega_{\delta_m}$ and any $m\geqq m_4$ we have

(3. 4) $$\|J_m(\alpha)-J_m(\hat{\alpha})\|\leqq M_3\|\alpha-\hat{\alpha}\|\leqq M_3\delta_m\leqq M_3\delta_1\leqq\kappa/M'.$$

Finally by the inequalities (3. 1) and (3. 2) we have

(3. 5) $$\frac{M'\|\rho\|}{1-\kappa} \leqq [M'M_1M_4/(1-\kappa)]\sigma_1(m) \leqq \delta_m.$$

The expressions (3. 3), (3. 4) and (3. 5) show that the conditions (1), (2) and (3) are fulfilled.

Thus by the Lemma in the section 0 we see that the determining equation (2. 3) has one and only one solution $\alpha=\bar{\alpha}$ in the domain Ω_{δ_m} satisfying

$$\det J_m(\bar{\alpha}) \neq 0$$

and

(3. 6) $$\|\bar{\alpha}-\hat{\alpha}\| \leqq \frac{M'\|\rho\|}{1-\kappa} \leqq [M'M_1M_4/(1-\kappa)]\sigma_1(m).$$

If we put

$$\bar{\alpha}=(\bar{a}_0, \bar{a}_1, \cdots\cdots, \bar{a}_m) \qquad \text{and} \qquad \bar{u}_m(t)=\sum_{n=0}^{m} e_n \bar{a}_n T_n(t),$$

then $\bar{u}_m(t)$ is a Chebyshev approximation and satisfies for $m \geqq m_4$

$$\|\bar{u}_m-\hat{u}\|_n \leqq \|\bar{u}_m-\hat{u}_m\|_n + \|\hat{u}_m-\hat{u}\|_n \leqq \sqrt{2m+1}\|\bar{\alpha}-\hat{\alpha}\| + M_1\sigma(m)$$
$$\leqq [M'M_1M_4/(1-\kappa)]\sqrt{2m+1}\,\sigma_1(m) + M_1\sigma(m)$$

from the inequalities (1. 6) and (3. 6). The functions $\sqrt{2m+1}\,\sigma_1(m)$ and $\sigma(m)$ are equal to $O(m^{-1/2})$ as $m\to\infty$. This proves the existence of a Chebyshev approximation $\bar{u}_m(t)$ being convergent uniformly to the solution $\hat{u}(t)$ as $m\to\infty$.

4. Proof of Theorem 2

Let $u=\hat{u}(t)$ be an isolated solution of the equation (0. 1). For any positive number ε we choose a number m_0 such that for any $m \geqq m_0, \sigma_1(m)=1/(m+1)<\varepsilon$ and suppose that for any $m \geqq m_0$ there are two Chebyshev approximations

(4. 1) $\quad u=\bar{u}_m(t) \quad$ and $\quad u=\bar{u}_m'(t)$

satisfying

(4. 2) $\quad \|\bar{u}_m-\hat{u}\|_n \leqq \varepsilon \quad$ and $\quad \|\bar{u}_m'-\hat{u}\|_n \leqq \varepsilon.$

Hence the two Chebyshev approximations (4. 1) lie in the domain

$$D_\varepsilon=\{u \mid |u-\hat{u}(t)|<\varepsilon \text{ for some } t\in J\}\subset D$$

for any $t\in J$. Let us put $v_m(t)=\bar{u}_m(t)-\bar{u}_m'(t)$. By the definition of the Chebyshev approximations (4. 1) we have

$$(4.\ 3) \qquad v_m(t)=P_m\int_{-1}^{1}[K(t,s,\bar{u}_m(s))-K(t,s,\bar{u}_m'(s))]ds$$

$$=\int_{-1}^{1}\frac{\partial K}{\partial u}(t,s,\hat{u}(s))v_m(s)ds+R(t),$$

where

$$R(t)=P_m\int_{-1}^{1}\int_{0}^{1}[\frac{\partial K}{\partial u}(t,s,\bar{u}_m^\theta(s))-\frac{\partial K}{\partial u}(t,s,\hat{u}(s))]v_m(s)d\theta ds$$

$$-(I-P_m)\int_{-1}^{1}\frac{\partial K}{\partial u}(t,s,\hat{u}(s))v_m(s)ds$$

and

$$\bar{u}_m^\theta(t)=\bar{u}_m'(t)+\theta[\bar{u}_m(t)-\bar{u}_m'(t)].$$

Noting the fact that $\bar{u}_m^\theta(t)\in D_\varepsilon$ for any $\theta\in[0,1]$ and any $t\in J$, it is easy to see from the Parseval's equality (1. 4) and Schwarz's inequality that

$$\|R_m\|_q \leq M_3 \varepsilon \|v_m\|_q + \sigma_1(m) M_2 \|v_m\|_q \leq (M_3+M_2)\varepsilon \|v_m\|_q,$$

where M_2 and M_3 are the constant numbers defined in the section 2. Using the theory for linear integral equations of Fredholm type for the equation (4. 3) and the definition of the isolated solution $u=\hat{u}(t)$, we obtain

$$\|v_m\|_q \leq M \|R_m\|_q \leq M(M_3+M_2)\varepsilon \|v_m\|_q,$$

where M is the constant number used in the section 2. Since ε is arbitrary, the inequality above implies that

$$\|v_m\|_q = 0.$$

From the Parseval's equality (1. 5) for the finite Chebyshev series $v_m(t)$ it follows that

$$v_m(t)=0 \qquad \text{for any } t \in J,$$

that is,

$$\bar{u}_m(t)=\bar{u}_m'(t) \qquad \text{for any } t \in J.$$

This proves the uniqueness of Chebyshev approximations and hence completes the proof of Theorem 2.

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