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OBITUARY

Minoru URABE

(1912-1975)

Professor Minoru Urabe died of lung cancer on September 4, 1975. He was 62 years of age.

Minoru Urabe was born in Kobe on December 2, 1912. He graduated from the Hiroshima University of Science and Literature (now Hiroshima University) in 1940, continued his studies in mathematics and received the doctorate from the same University in 1953.

In 1946 he assumed a post in the mathematical teaching staff at Hiroshima University and became Professor of Mathematics there in 1952. He was appointed Professor at Kyushu University in 1963, Professor at Kyoto University (Research Institute of Mathematical Sciences) in 1966, and in 1971 he returned to Kyushu University as Professor, thereafter holding this post until his death.

His researches began with geometry and subsequently extended to functional equations, ordinary differential equations, numerical analysis and nonlinear oscillations. The paper "Galerkin's Procedure for Nonlinear Periodic Systems" (Arch. Rational Mech. Anal., 20(1965), 120-152) and the book "Nonlinear Autonomous Oscillations-Analytical Theory" (Academic Press, New York, 1967) are among his most fundamental and well-known publications.

The outstanding research work and scholarly attitude of Minoru Urabe constituted a source of great stimulation and encouragement to his friends, colleagues and students, who will always remember him with affection and gratitude.

Yoshitane SHINOHARA

REFERENCE

Professoional career and works of Late Professor Minoru URABE, (including his portrait, and complete list of his publications), Memoirs of the Faculty of Science, Kyushu University Series A, Mathematics, Vol. 30, No.2, 1976, p.157-168.

CONTENTS

Energy Estimates for the Solution of Hyperbolic Equations by a Finite Element Mass Scheme

Kazuo ISHIHARA*

Summary

The solution of the initial boundary value problem for hyperbolic equations is approximated by the finite element method with the generalized mixed mass scheme presented in the previous papers([4],[5]). The stability condition is obtained and the rate of convergence is established for the approximation. Numerical experiments are also performed.

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1. Introduction

This paper concerns the finite element schemes applied to the initial boundary value problem for hyperbolic type:

$$
\begin{aligned}
&\partial^2 u/\partial t^2 = \Delta u + f(x,t) && x \in \Omega, \quad 0 < t \leq T, \\
&u = 0 && \text{on } \Gamma, \quad 0 < t \leq T, \\
&u(x,0) = u_0(x) && x \in \Omega, \\
&\frac{\partial}{\partial t} u(x,0) = v_0(x) && x \in \Omega.
\end{aligned} \tag{1}
$$

Here f, u_0 and v_0 are given smooth functions, Δ is the Laplacian operator and $x=(x_1,x_2,\cdots,x_m)$ is a point of a bounded domain Ω in the m-dimensional Euclidean space R^m with the smooth boundary Γ.

Let $L_2(\Omega)$ be the usual real space of square integrable functions on Ω. The scalar product and the norm on $L_2(\Omega)$ are denoted by $(\cdot,\cdot)$ and $\|\cdot\|$, respectively. $H^1(\Omega)$ denotes the real 1-st order Sobolev space. $H_0^1(\Omega)$ is the set defined by

$$H_0^1(\Omega) = \{u \in H^1(\Omega) : u=0 \ \text{ on } \ \Gamma\}.$$

The weak solution of (1) is defined as a function $u \in H_0^1(\Omega)$, which satisfies the weak form:

$$(\partial^2 u/\partial t^2, v) + a(u,v) = (f,v), \quad 0 < t \leq T \quad \text{for each} \quad v \in H_0^1(\Omega) \tag{2}$$

where $a(u,v)$ is given by

$$a(u,v) = \int_\Omega \{\Sigma_{i=1}^m \partial u/\partial x_i \cdot \partial v/\partial x_i\} dx_1 dx_2 \cdots dx_m.$$

To introduce the step-by-step methods, we set $u^n = u(x, n\Delta t)$, $n=0,1,2,\cdots,p$. Here Δt is the time increment and $p\cdot\Delta t = T$. We apply to (2) the consistent mass(CM) scheme and the lumped mass(LM) scheme with piecewise linear polynomials. Then the corresponding equations may be written in the following forms by the step-by-step methods with a parameter $\beta(\geq 0)$:

$$M_1 D_t D_{\bar{t}} \hat{V}^n + K\hat{V}^n + \beta\Delta t^2 K D_t D_{\bar{t}} \hat{V}^n = \hat{F}^n \qquad \text{for the CM scheme} \qquad (3)$$

$$M_2 D_t D_{\bar{t}} \bar{V}^n + K\bar{V}^n + \beta\Delta t^2 K D_t D_{\bar{t}} \bar{V}^n = \bar{F}^n \qquad \text{for the LM scheme} \qquad (4)$$

where $\hat{V}^n$ and $\bar{V}^n$ are unknown vectors, D_t and $D_{\bar{t}}$ are forward and backward difference operators in time defined by

$$D_t V^n = (V^{n+1} - V^n)/\Delta t, \qquad D_{\bar{t}} V^n = (V^n - V^{n-1})/\Delta t,$$

K is the stiffness matrix, M_1 is the CM matrix, M_2 is the LM matrix, and $\hat{F}^n$, $\bar{F}^n$ are known vectors.

In the previous papers([4],[5]), the author presented the generalized mixed mass(GMM) scheme for the eigenvalue and parabolic problems. In this paper, we propose similarly the GMM scheme for the hyperbolic problem. The equation for the GMM scheme with parameters α and $\beta(0 \leq \alpha \leq 1,\ \beta \geq 0)$ is as follows:

$$\{\alpha M_1 + (1-\alpha)M_2\} D_t D_{\bar{t}} V^n + KV^n + \beta\Delta t^2 K D_t D_{\bar{t}} V^n = \alpha\hat{F}^n + (1-\alpha)\bar{F}^n \qquad (5)$$

$$n = 1, 2, \cdots, p-1.$$

For this scheme, we can derive the stability condition in the L_2 sense and establish the error estimates. The GMM scheme includes the CM scheme(α=1) and the LM scheme(α=0) as its special cases. Finally some numerical experiments are performed.

2. Stability Condition

Throughout this paper, we will use the same notations as the previous paper[5]. It is assumed that the domain Ω is the convex polygon and the solution u of (1) satisfies certain smoothness condition. Let T^h be a triangulation of the domain as follows:

$$\bar{\Omega} = \bigcup_{k=1}^{N} \bar{\Delta}_k, \qquad \Delta_i \cap \Delta_j = \emptyset, \quad (i \neq j).$$

Here $\Delta_k (k=1,2,\cdots,N)$ are disjoint non-degenerate m-simplices such that any one of its faces is either a face of another m-simplex or else is a portion of Γ, and h is the largest side length of all the m-simplices of T^h. By $P_i, 1 \leq i \leq n$, (or $P_i, n+1 \leq i \leq n+J$) we denote the vertices of the triangulation T^h which belong to Ω(or Γ).

We now define the lumped mass region $B(P_i)$ corresponding to the vertex P_i with respect to T^h. Let $b_0=P_i$, $b_1,\cdots, b_m$ be the vertices of some m-simplex Δ_k of T^h. We define the barycentric coordinate λ_i corresponding to the vertex $b_i (0 \leq i \leq m)$. Then the barycentric subdivision B_i^k of Δ_k corresponding to P_i is defined by

$$B_i^k = \{x: \tfrac{1}{2} < \lambda_0(x)/(\lambda_0(x)+\lambda_j(x)) \leq 1 \quad \text{for any } j=1,\cdots,m\}.$$

The lumped mass region $B(P_i)$ is the union of B_i^k having P_i as its vertex. $\hat{\phi}_i \in C(\bar{\Omega})$ and $\bar{\phi}_i (i=1,2,\cdots,n+J)$ stand the functions which satisfy the relations:

$$\hat{\phi}_i(P_j)=\delta_{ij}, \quad (1\leq i,j\leq n+J),$$

$$\hat{\phi}_i \text{ is linear for each m-simplex } \Delta\in T^h \quad (1\leq i\leq n+J),$$

$$\bar{\phi}_i(P)=\begin{cases} 1 & P\in B(P_i) \\ 0 & P\notin B(P_i) \end{cases} \quad (1\leq i\leq n+J)$$

where δ_{ij} is Kronecker's delta. Define finite dimensional spaces $X^h(\subset L_2(\Omega))$, X_0^h, $Y^h(\subset H^1(\Omega))$ and Y_0^h as follows:

$$X^h=\mathrm{Span}[\bar{\phi}_1,\bar{\phi}_2,\cdots,\bar{\phi}_{n+J}],$$

$$X_0^h=\{\bar{\phi}: \quad \bar{\phi}\in X^h, \quad \bar{\phi}=0 \quad \text{on} \quad \Gamma\},$$

$$Y^h=\mathrm{Span}[\hat{\phi}_1,\hat{\phi}_2,\cdots,\hat{\phi}_{n+J}],$$

$$Y_0^h=\{\hat{\phi}: \quad \hat{\phi}\in Y^h, \quad \hat{\phi}=0 \quad \text{on} \quad \Gamma\}.$$

Every $\bar{\phi}\in X^h$ and $\hat{\phi}\in Y^h$ can be uniquely determined as

$$\bar{\phi} = \Sigma_{i=1}^{n+J}\alpha_i\bar{\phi}_i,$$

$$\hat{\phi} = \Sigma_{i=1}^{n+J}\beta_i\hat{\phi}_i$$

where α_i and β_i are nodal values. Two functions $\bar{\phi}$ and $\hat{\phi}$ are called associative and denoted by $\bar{\phi}\sim\hat{\phi}$, if they have a common nodal value at each vertex. Following to Ciarlet-Raviart[1] and Fujii[3], we also introduce the parameters κ and σ which are associated with the triangulation T^h. We denote by κ the minimum perpendicular length of all the m-simplices of T^h. Let $\lambda_i(0\leq i\leq m+1)$ be the barycentric coordinate of a point $x\in\Delta(\in T^h)$ with respect to the vertex P_i. We associate the parameter

$$\sigma_\Delta = \max_{i\neq j}\{\cos(D\lambda_i,D\lambda_j)\}$$

with

$$D\lambda_i=(\partial\lambda_i/\partial x_1,\cdots,\partial\lambda_i/\partial x_m),\quad 1\leq i\leq m+1,$$

$$\cos(D\lambda_i,D\lambda_j)=\frac{\langle D\lambda_i,D\lambda_j\rangle}{|D\lambda_i|\cdot|D\lambda_j|},\quad 1\leq i,j\leq m+1,$$

where $\langle\cdot,\cdot\rangle$ and $|\cdot|$ respectively denote the Euclidean scalar product and Euclidean norm in R^m. Then σ is defined by

$$\sigma=\max_{\Delta\in T^h}\sigma_\Delta.$$

An acute triangulation satisfies the condition $\sigma\leq 0$([1],[3]). We will use the following notations and definitions:

$K=\{a(\hat{\phi}_i,\hat{\phi}_j)\}$	$(1\leq i,j\leq n+J)$	stiffness matrix,
$M_1=\{(\hat{\phi}_i,\hat{\phi}_j)\}$	$(1\leq i,j\leq n+J)$	CM matrix,
$M_2=\{(\bar{\phi}_i,\bar{\phi}_j)\}$	$(1\leq i,j\leq n+J)$	LM matrix,
$M_3=\alpha M_1+(1-\alpha)M_2$	$(0\leq\alpha\leq 1)$	GMM matrix,
$\hat{F}=\{(f,\hat{\phi}_i)\}$	$(1\leq i\leq n+J)$,	
$\bar{F}=\{(f,\bar{\phi}_i)\}$	$(1\leq i\leq n+J)$,	

$$A_m=\begin{cases}2 & (\sigma\leq 0)\\ m+1 & (\sigma>0).\end{cases}$$

The solutions $\{\hat{v}^n,\bar{v}^n\}$ $(n=0,1,\cdots,p)$ of the GMM scheme are defined with parameters α and β $(0\leq\alpha\leq 1,\ \beta\geq 0)$ as follows:

$$\begin{aligned}\alpha(D_tD_{\bar{t}}\hat{v}^n,\hat{\phi})+(1-\alpha)(D_tD_{\bar{t}}\bar{v}^n,\bar{\phi})+a(\hat{v}^n,\hat{\phi})+\beta\Delta t^2a(D_tD_{\bar{t}}\hat{v}^n,\hat{\phi})\\ =\alpha(f^n,\hat{\phi})+(1-\alpha)(f^n,\bar{\phi}),\qquad n=1,2,\cdots,p-1,\end{aligned}\tag{6}$$

$$\hat{v}^n\in Y_0^h,\ \bar{v}^n\in X_0^h,\ \hat{v}^n\sim\bar{v}^n\quad\text{for each }\hat{\phi}\in Y_0^h,\ \bar{\phi}\in X_0^h,\ \hat{\phi}\sim\bar{\phi},$$

where

$$\hat{v}^0=\Sigma_{i=1}^{n+J}u_0(P_i)\hat{\phi}_i,\qquad \bar{v}^0=\Sigma_{i=1}^{n+J}u_0(P_i)\bar{\phi}_i,$$

$$\hat{v}^1=\Sigma_{i=1}^{n+J}\{u_0(P_i)+\Delta t v_0(P_i)\}\hat{\phi}_i, \quad \bar{v}^1=\Sigma_{i=1}^{n+J}\{u_0(P_i)+\Delta t v_0(P_i)\}\bar{\phi}_i.$$

This scheme is equivalent to the matrix expression (5). The GMM scheme includes the LM scheme($\alpha=0$) and the CM scheme($\alpha=1$) as its special cases. We assume that $0<\alpha<1$. Our results are valid for $\alpha=0$ and $\alpha=1$. In these cases we can obtain the similar results discussed in [2].

Now we shall derive the stability condition. It is well known that the solution u of (1) satisfies the following energy inequality:

$$\begin{aligned}&\|\partial u/\partial t\|^2(t)+\Sigma_{i=1}^{m}\|\partial u/\partial x_i\|^2(t)\\&\quad\leq c_1(\|v_0\|^2+\Sigma_{i=1}^{m}\|\partial u_0/\partial x_i\|^2+\int_0^t\|f\|^2dt), \qquad 0<t\leq T\end{aligned} \tag{7}$$

where c_1 is a positive constant. We say that the GMM scheme is stable if the solution $\{\hat{v}^n,\bar{v}^n\}$ of (6) satisfies the energy inequality, anagolous to (7), that is,

$$\begin{aligned}&\alpha\|D_{\bar{t}}\hat{v}^n\|^2+(1-\alpha)\|D_{\bar{t}}\bar{v}^n\|^2+\Sigma_{i=1}^{m}\|\partial\hat{v}^n/\partial x_i\|^2\\&\quad\leq c_2\{\alpha\|D_t\hat{v}^0\|^2+(1-\alpha)\|D_t\bar{v}^0\|^2+\Sigma_{i=1}^{m}\|\partial\hat{v}^0/\partial x_i\|^2+\Sigma_{i=1}^{n-1}\Delta t\|f^i\|^2\},\\&\qquad n=2,3,\cdots,p\end{aligned}$$

where c_2 is a positive constant. The stability condition is derived using the following lemmas.

Lemma 1. For any $\hat{w}\in Y^h$ and $\bar{w}\in X^h$ $(\hat{w}\sim\bar{w})$, it holds that

$$\Sigma_{i=1}^{m}\|\partial\hat{w}/\partial x_i\|^2\leq A\{\alpha\|\hat{w}\|^2+(1-\alpha)\|\bar{w}\|^2\}$$

where

$$A=\frac{A_m(m+1)(m+2)}{\kappa^2\{m+2-(m+1)\alpha\}}.$$

Proof. Fujii([3]) has shown the following results:

$$a(\hat{w},\hat{w})\leq\frac{A_m(m+1)(m+2)}{\kappa^2}\|\hat{w}\|^2, \qquad a(\hat{w},\hat{w})\leq\frac{A_m(m+1)}{\kappa^2}\|\bar{w}\|^2.$$

Combining these two inequalities yields the desired statement.

Lemma 2. Let x_n be the nonnegative sequence $(n=1,2,\cdots,p)$. If $\tilde{c} \geq 0$, $0 \leq t < 1$ and $x_n \leq \tilde{c} + \Sigma_{i=1}^{n} t x_i$, $n=1,2,\cdots,p$, then,

$$x_n \leq \tilde{c}/(1-t)^n,$$

and

$$\Sigma_{i=1}^{n} t x_i \leq \tilde{c}\{1/(1-t)^n - 1\}, \qquad n=1,2,\cdots,p.$$

Proof. This lemma is easily proved by induction.

Theorem 1. The GMM scheme is unconditionally stable if $\beta \geq 1/4$, or stable under the condition

$$\frac{\Delta t}{\kappa} < \sqrt{\frac{m+2-(m+1)\alpha}{A_m(m+1)(m+2)}} \cdot \frac{2}{\sqrt{1-4\beta}}$$

if $0 \leq \beta < 1/4$.

Proof. Choosing $\hat{\phi} = D_t\hat{v}^n + D_{\bar{t}}\hat{v}^n$, $\bar{\phi} = D_t\bar{v}^n + D_{\bar{t}}\bar{v}^n$ in (6), multiplying Δt and summing from $n=1$ to $n=r-1$, we have

$$\alpha(\|D_{\bar{t}}\hat{v}^r\|^2 - \|D_t\hat{v}^0\|^2) + (1-\alpha)(\|D_{\bar{t}}\bar{v}^r\|^2 - \|D_t\bar{v}^0\|^2) + a(\hat{v}^r,\hat{v}^r) - a(\hat{v}^0,\hat{v}^0)$$

$$-\Delta t\{a(\hat{v}^r, D_{\bar{t}}\hat{v}^r) + a(\hat{v}^0, D_t\hat{v}^0)\} + \beta\Delta t^2\{a(D_{\bar{t}}\hat{v}^r, D_{\bar{t}}\hat{v}^r) - a(D_t\hat{v}^0, D_t\hat{v}^0)\}$$

$$= \Sigma_{n=1}^{r-1}\Delta t\{\alpha(f^n, D_t\hat{v}^n + D_{\bar{t}}\hat{v}^n) + (1-\alpha)(f^n, D_t\bar{v}^n + D_{\bar{t}}\bar{v}^n)\}$$

$$\leq \alpha\Sigma_{n=1}^{r-1}\Delta t\|f^n\|^2 + \alpha\Sigma_{n=1}^{r}\Delta t\|D_{\bar{t}}\hat{v}^n\|^2 + (1-\alpha)\Sigma_{n=1}^{r-1}\Delta t\|f^n\|^2 + (1-\alpha)\Sigma_{n=1}^{r}\Delta t\|D_{\bar{t}}\bar{v}^n\|^2$$

$$= \Sigma_{n=1}^{r-1}\Delta t\|f^n\|^2 + \Sigma_{n=1}^{r}\Delta t\{\alpha\|D_{\bar{t}}\hat{v}^n\|^2 + (1-\alpha)\|D_{\bar{t}}\bar{v}^n\|^2\}.$$

Here we have used the following identities:

$$\Sigma_{n=1}^{r-1}\Delta t(D_t D_{\bar{t}}\hat{v}^n, D_t\hat{v}^n + D_{\bar{t}}\hat{v}^n) = \|D_{\bar{t}}\hat{v}^r\|^2 - \|D_t\hat{v}^0\|^2,$$

$$\Sigma_{n=1}^{r-1}\Delta t a(D_t D_{\bar{t}}\hat{v}^n, D_t\hat{v}^n + D_{\bar{t}}\hat{v}^n) = a(D_{\bar{t}}\hat{v}^r, D_{\bar{t}}\hat{v}^r) - a(D_t\hat{v}^0, D_t\hat{v}^0),$$

$$\Sigma_{n=1}^{r-1}\Delta t a(\hat{v}^n, D_t\hat{v}^n + D_{\bar{t}}\hat{v}^n) = a(\hat{v}^r,\hat{v}^r) - a(\hat{v}^0,\hat{v}^0) - \Delta t\{a(\hat{v}^r, D_{\bar{t}}\hat{v}^r) + a(\hat{v}^0, D_t\hat{v}^0)\}.$$

Therefore, for an arbitrary number $\varepsilon > 0$, we obtain

$$\alpha\|D_{\bar{t}}\hat{v}^r\|^2+(1-\alpha)\|D_{\bar{t}}\bar{v}^r\|^2+a(\hat{v}^r,\hat{v}^r)+\beta\Delta t^2 a(D_{\bar{t}}\hat{v}^r,D_{\bar{t}}\hat{v}^r)$$

$$\leq \alpha\|D_t\hat{v}^0\|^2+(1-\alpha)\|D_t\bar{v}^0\|^2+a(\hat{v}^0,\hat{v}^0)+\beta\Delta t^2 a(D_t\hat{v}^0,D_t\hat{v}^0)+a(\hat{v}^r,\Delta t D_{\bar{t}}\hat{v}^r)+$$

$$a(\hat{v}^0,\Delta t D_t\hat{v}^0)+\Sigma_{n=1}^{r-1}\Delta t\|f^n\|^2+\Sigma_{n=1}^{r}\Delta t\{\alpha\|D_{\bar{t}}\hat{v}^n\|^2+(1-\alpha)\|D_{\bar{t}}\bar{v}^n\|^2\}$$

$$\leq \alpha\|D_t\hat{v}^0\|^2+(1-\alpha)\|D_t\bar{v}^0\|^2+a(\hat{v}^0,\hat{v}^0)+\beta\Delta t^2 a(D_t\hat{v}^0,D_t\hat{v}^0)+\frac{\varepsilon}{2}a(\hat{v}^r,\hat{v}^r)+$$

$$\frac{\Delta t^2}{2\varepsilon}a(D_{\bar{t}}\hat{v}^r,D_{\bar{t}}\hat{v}^r)+\frac{\varepsilon}{2}a(\hat{v}^0,\hat{v}^0)+\frac{\Delta t^2}{2\varepsilon}a(D_t\hat{v}^0,D_t\hat{v}^0)+\Sigma_{n=1}^{r-1}\Delta t\|f^n\|^2$$

$$+\Sigma_{n=1}^{r}\Delta t\{\alpha\|D_{\bar{t}}\hat{v}^n\|^2+(1-\alpha)\|D_{\bar{t}}\bar{v}^n\|^2\}.$$

From Lemma 1, this may be written as

$$\alpha\|D_{\bar{t}}\hat{v}^r\|^2+(1-\alpha)\|D_{\bar{t}}\bar{v}^r\|^2+(1-\frac{\varepsilon}{2})a(\hat{v}^r,\hat{v}^r)$$

$$\leq \alpha\|D_t\hat{v}^0\|^2+(1-\alpha)\|D_t\bar{v}^0\|^2+(1+\frac{\varepsilon}{2})a(\hat{v}^0,\hat{v}^0)+(\frac{1}{2\varepsilon}-\beta)\Delta t^2 a(D_{\bar{t}}\hat{v}^r,D_{\bar{t}}\hat{v}^r)+$$

$$\Delta t^2(\frac{1}{2\varepsilon}+\beta)a(D_t\hat{v}^0,D_t\hat{v}^0)+\Sigma_{n=1}^{r-1}\Delta t\|f^n\|^2+\Sigma_{n=1}^{r}\Delta t\{\alpha\|D_{\bar{t}}\hat{v}^n\|^2+(1-\alpha)\|D_{\bar{t}}\bar{v}^n\|^2\}$$

$$\leq \alpha\|D_t\hat{v}^0\|^2+(1-\alpha)\|D_t\bar{v}^0\|^2+(1+\frac{\varepsilon}{2})a(\hat{v}^0,\hat{v}^0)+$$

$$\max\{0,\frac{1}{2\varepsilon}-\beta\}\Delta t^2 A\{\alpha\|D_{\bar{t}}\hat{v}^r\|^2+(1-\alpha)\|D_{\bar{t}}\bar{v}^r\|^2\}+$$

$$(\frac{1}{2\varepsilon}+\beta)\Delta t^2 A\{\alpha\|D_t\hat{v}^0\|^2+(1-\alpha)\|D_t\bar{v}^0\|^2\}+\Sigma_{n=1}^{r-1}\Delta t\|f^n\|^2+$$

$$\Sigma_{n=1}^{r}\Delta t\{\alpha\|D_{\bar{t}}\hat{v}^n\|^2+(1-\alpha)\|D_{\bar{t}}\bar{v}^n\|^2+a(\hat{v}^n,\hat{v}^n)\},$$

that is,

$$[1-\max\{0,\frac{1}{2\varepsilon}-\beta\}\Delta t^2 A]\{\alpha\|D_{\bar{t}}\hat{v}^r\|^2+(1-\alpha)\|D_{\bar{t}}\bar{v}^r\|^2\}+(1-\frac{\varepsilon}{2})a(\hat{v}^r,\hat{v}^r)$$

$$\leq [1+(\frac{\varepsilon}{2}+\beta)\Delta t^2 A]\{\alpha\|D_t\hat{v}^0\|^2+(1-\alpha)\|D_t\bar{v}^0\|^2\}+(1+\frac{\varepsilon}{2})a(\hat{v}^0,\hat{v}^0)+$$

$$\Sigma_{n=1}^{r-1}\Delta t\|f^n\|^2+\Sigma_{n=1}^{r}\Delta t\{\alpha\|D_{\bar{t}}\hat{v}^n\|^2+(1-\alpha)\|D_{\bar{t}}\bar{v}^n\|^2+a(\hat{v}^n,\hat{v}^n)\}.$$

Then, we can obtain the following inequality

$$\alpha\|D_{\bar t}\hat v^r\|^2+(1-\alpha)\|D_{\bar t}\bar v^r\|^2+a(\hat v^r,\hat v^r)$$

$$\leq C\{\alpha\|D_t\hat v^0\|^2+(1-\alpha)\|D_t\bar v^0\|^2+a(\hat v^0,\hat v^0)+\Sigma_{n=1}^{r-1}\Delta t\|f^n\|^2\}$$

with some positive constant C, from Lemma 2, if

$$\begin{aligned} &1-\max\{0,\frac{1}{2\varepsilon}-\beta\}\Delta t^2 A>0,\\ &1-\frac{\varepsilon}{2}>0 \end{aligned} \tag{8}$$

holds simultaneously. If $\beta>1/4$, (8) is satisfied for any Δt and κ by choosing $\varepsilon=\frac{1}{2\beta}$. If $0<\beta\leq 1/4$, (8) is satisfied when the quadratic equation in ε

$$1-(\frac{1}{2\varepsilon}-\beta)\frac{\Delta t^2 A_m(m+1)(m+2)}{\kappa^2\{m+2-(m+1)\alpha\}}=1-\frac{\varepsilon}{2}$$

has a positive root $\varepsilon(0<\varepsilon<2)$. This is satisfied for any Δt and κ if $\beta=1/4$, or for the condition

$$\frac{\Delta t}{\kappa}<\sqrt{\frac{m+2-(m+1)\alpha}{A_m(m+1)(m+2)}}\cdot\frac{2}{\sqrt{1-4\beta}}$$

if $0<\beta<1/4$. This completes the proof.

3. Rate of Convergence

This section gives the rate of convergence for the GMM scheme. In the sequel, $C_1, C_2, \cdots$ are positive constants which are independent of h and Δt. Let $\hat{u} \in Y_0^h$ and $\bar{u} \in X_0^h$ be the associative interpolated functions which coincide with u at each vertex. Then it is well known that

$$\| u-\hat{u} \|^2 + a(u-\hat{u}, u-\hat{u}) \le \hat{C}_1 h^2, \tag{9}$$

$$\| u-\bar{u} \|^2 + a(u-\hat{u}, u-\hat{u}) \le \hat{C}_2 h^2 \tag{10}$$

where $\hat{C}_1$ and $\hat{C}_2$ are positive constants which are independent of h([2],[6]).

On the other hand, from the expansion we have

$$D_t D_{\bar{t}} u^n = \partial^2 u/\partial t^2 + \beta \Delta t^2 D_t D_{\bar{t}} \partial^2 u^n/\partial t^2 + \Delta t w^n$$

where w^n is bounded. Therefore, from (2) it holds that

$$\begin{aligned}(D_t D_{\bar{t}} u^n, \hat{\phi}) &= (\partial^2 u^n/\partial t^2, \hat{\phi}) + \beta\Delta t^2 (D_t D_{\bar{t}} \partial^2 u^n/\partial t^2, \hat{\phi}) + (\Delta t w^n, \hat{\phi}) \\ &= -a(u^n, \hat{\phi}) + (f^n, \hat{\phi}) - \beta\Delta t^2 a(D_t D_{\bar{t}} u^n, \hat{\phi}) + \beta\Delta t^2 (D_t D_{\bar{t}} f^n, \hat{\phi}) \\ &\quad + (\Delta t w^n, \hat{\phi}) \qquad \text{for each } \hat{\phi} \in Y_0^h (\subset H_0^1(\Omega)).\end{aligned}$$

Then we have

$$\begin{aligned}&\alpha(D_t D_{\bar{t}} \hat{u}^n, \hat{\phi}) + (1-\alpha)(D_t D_{\bar{t}} \bar{u}^n, \bar{\phi}) + a(\hat{u}^n, \hat{\phi}) + \beta\Delta t^2 a(D_t D_{\bar{t}} \hat{u}^n, \hat{\phi}) \\ &= \alpha(D_t D_{\bar{t}} \hat{u}^n, \hat{\phi}) + (1-\alpha)(D_t D_{\bar{t}} \bar{u}^n, \bar{\phi}) + a(\hat{u}^n, \hat{\phi}) + \beta\Delta t^2 a(D_t D_{\bar{t}} \hat{u}^n, \hat{\phi}) - (D_t D_{\bar{t}} u^n, \hat{\phi}) \\ &\quad -a(u^n, \hat{\phi}) - \beta\Delta t^2 a(D_t D_{\bar{t}} u^n, \hat{\phi}) + (f^n, \hat{\phi}) + \beta\Delta t^2 (D_t D_{\bar{t}} f^n, \hat{\phi}) + (\Delta t w^n, \hat{\phi}) \\ &= \alpha(D_t D_{\bar{t}}(\hat{u}^n - u^n), \hat{\phi}) + (1-\alpha)(D_t D_{\bar{t}}(\bar{u}^n - u^n), \bar{\phi}) + (1-\alpha)(D_t D_{\bar{t}} u^n, \bar{\phi} - \hat{\phi}) + \\ &\quad a(\hat{u}^n - u^n, \hat{\phi}) + \beta\Delta t^2 a(D_t D_{\bar{t}}(\hat{u}^n - u^n), \hat{\phi}) + \alpha(f^n, \hat{\phi}) + (1-\alpha)(f^n, \hat{\phi}) - \\ &\quad (1-\alpha)(f^n, \bar{\phi}) + (1-\alpha)(f^n, \bar{\phi}) + \beta\Delta t^2 (D_t D_{\bar{t}} f^n, \hat{\phi}) + (\Delta t w^n, \hat{\phi})\end{aligned} \tag{11}$$

for each $\hat{\phi}\in Y_0^h$, $\bar{\phi}\in X_0^h$, $\hat{\phi}\sim\bar{\phi}$.

Putting $\hat{e}^n=\hat{u}^n-\hat{v}^n$, $\bar{e}^n=\bar{u}^n-\bar{v}^n$, and substracting (6) from (11), we obtain

$$\alpha(D_tD_{\bar{t}}\hat{e}^n,\hat{\phi})+(1-\alpha)(D_tD_{\bar{t}}\bar{e}^n,\bar{\phi})+a(\hat{e}^n,\hat{\phi})+\beta\Delta t^2a(D_tD_{\bar{t}}\hat{e}^n,\hat{\phi})$$

$$=\alpha(D_tD_{\bar{t}}(\hat{u}^n-u^n),\hat{\phi})+(1-\alpha)(D_tD_{\bar{t}}(\bar{u}^n-u^n),\bar{\phi})+(1-\alpha)(D_tD_{\bar{t}}u^n,\bar{\phi}-\hat{\phi})$$

$$+a(\hat{u}^n-u^n,\hat{\phi})+\beta\Delta t^2a(D_tD_{\bar{t}}(\hat{u}^n-u^n),\hat{\phi})-(1-\alpha)(f^n,\bar{\phi}-\hat{\phi})$$

$$+\beta\Delta t^2(D_tD_{\bar{t}}f^n,\hat{\phi})+(\Delta tw^n,\hat{\phi})$$

for each $\hat{\phi}\in Y_0^h$, $\bar{\phi}\in X_0^h$, $\hat{\phi}\sim\bar{\phi}$.

Before stating our results, we mention some lemmas which are useful.

Lemma 3.(Fujii[2]) For any $\hat{w}\in Y^h$ and $\bar{w}\in X^h$ $(\hat{w}\sim\bar{w})$, there exists a constant c which is independent of h, such that

$$\|\hat{w}-\bar{w}\|^2\leq ch^2\Sigma_{i=1}^m\|\partial\hat{w}/\partial x_i\|^2.$$

Lemma 4. For any $\hat{w}\in Y^h$ and $\bar{w}\in X^h$ $(\hat{w}\sim\bar{w})$, it holds that

$$\|\hat{w}\|^2\leq\|\bar{w}\|^2\leq(m+2)\|\hat{w}\|^2.$$

Proof. Let Δ be an m-simplex of T^h. We put $\hat{w}=\Sigma_{i=1}^{m+1}w_i\hat{\phi}_i$, and

$\bar{w}=\Sigma_{i=1}^{m+1} w_i \bar{\phi}_i$. Then we have

$$\|\bar{w}\|_\Delta^2 = \int_\Delta \bar{w}^2 dx_1 \cdots dx_m = \frac{\mathrm{vol}(\Delta)}{m+1} \Sigma_{i=1}^{m+1} w_i^2,$$

$$\|\hat{w}\|_\Delta^2 = \int_\Delta \hat{w}^2 dx_1 \cdots dx_m = \frac{\mathrm{vol}(\Delta)}{(m+1)(m+2)} (2\Sigma_{i=1}^{m+1} w_i^2 + 2\Sigma_{i=1}^{m} \Sigma_{j=i+1}^{m+1} w_i w_j)$$

$$\geq \frac{\mathrm{vol}(\Delta)}{(m+1)(m+2)} \Sigma_{i=1}^{m+1} w_i^2 = \|\bar{w}\|_\Delta^2/(m+2),$$

$$\|\bar{w}\|_\Delta^2 - \|\hat{w}\|_\Delta^2 = \frac{\mathrm{vol}(\Delta)}{(m+1)(m+2)} \Sigma_{i=1}^{m} \Sigma_{j=i+1}^{m+1} (w_i - w_j)^2 \geq 0.$$

The proof is complete.

Lemma 5. For any $\hat{w}_0 \in Y_0^h$ and $\bar{w}_0 \in X_0^h$ $(\hat{w} \sim \bar{w})$, there exists a constant $\hat{C}$, which is independent of h, such that

$$\alpha\|\hat{w}_0\|^2 + (1-\alpha)\|\bar{w}_0\|^2 \leq \hat{C} a(\hat{w}_0, \hat{w}_0).$$

Proof. From Lemma 4 and Poincaré's inequality, we have

$$\alpha\|\hat{w}_0\|^2 + (1-\alpha)\|\bar{w}_0\|^2 \leq (m+2)\alpha\|\hat{w}_0\|^2 + (m+2)(1-\alpha)\|\hat{w}_0\|^2$$

$$= (m+2)\|\hat{w}_0\|^2 \leq (m+2)C_0 a(\hat{w}_0, \hat{w}_0) = \hat{C} a(\hat{w}_0, \hat{w}_0)$$

where C_0, $\hat{C}$ are positive constants. The proof is complete.

We now prove the following theorems which give the rate of convergence.

Theorem 2. Let $\{\hat{v}^r, \bar{v}^r\}$ be the solutions of (6). If the stability condition is satisfied, then, for sufficiently small Δt, there exists a constant $\bar{C}$, which is independent of h and Δt, such that

$$\alpha\|\hat{e}^r\|^2 + (1-\alpha)\|\bar{e}^r\|^2 + \alpha\|D_{\bar{t}}\hat{e}^r\|^2 + (1-\alpha)\|D_{\bar{t}}\bar{e}^r\|^2 + \Sigma_{i=1}^{m}\|\partial\hat{e}^r/\partial x_i\|^2$$

$$\leq \bar{C}(\Delta t^2 + h^2), \qquad r=2,3,\cdots,p,$$

where $\hat{e}^r = \hat{u}^r - \hat{v}^r$, $\bar{e}^r = \bar{u}^r - \bar{v}^r$.

Proof. Choosing $\hat{\phi}=D_t\hat{e}^n+D_{\bar{t}}\hat{e}^n$, $\bar{\phi}=D_t\bar{e}^n+D_{\bar{t}}\bar{e}^n$ in (12), multiplying Δt and summing from $n=1$ to $n=r-1$, we have

$$
\begin{aligned}
&\alpha\|D_{\bar{t}}\hat{e}^r\|^2+(1-\alpha)\|D_{\bar{t}}\bar{e}^r\|^2+a(\hat{e}^r,\hat{e}^r)\\
&=\alpha\|D_t\hat{e}^0\|^2+(1-\alpha)\|D_t\bar{e}^0\|^2+a(\hat{e}^0,\hat{e}^0)-\beta\Delta t^2a(D_{\bar{t}}\hat{e}^r,D_{\bar{t}}\hat{e}^r)+\beta\Delta t^2a(D_t\hat{e}^0,D_t\hat{e}^0)\\
&+a(e^r,\Delta tD_{\bar{t}}\hat{e}^r)+a(e^0,\Delta tD_t\hat{e}^0)+\alpha\Sigma_{n=1}^{r-1}\Delta t(D_tD_{\bar{t}}(\hat{u}^n-u^n),D_t\hat{e}^n+D_{\bar{t}}\hat{e}^n)+\\
&(1-\alpha)\Sigma_{n=1}^{r-1}\Delta t(D_tD_{\bar{t}}(\bar{u}^n-u^n),D_t\bar{e}^n+D_{\bar{t}}\bar{e}^n)+\Sigma_{n=1}^{r-1}\Delta ta(\hat{u}^n-u^n,D_t\hat{e}^n+D_{\bar{t}}\hat{e}^n)+\\
&\beta\Delta t^2\Sigma_{n=1}^{r-1}\Delta ta(D_tD_{\bar{t}}(\hat{u}^n-u^n),D_t\hat{e}^n+D_{\bar{t}}\hat{e}^n)+\\
&(1-\alpha)\Sigma_{n=1}^{r-1}\Delta t(D_tD_{\bar{t}}u^n,(D_t+D_{\bar{t}})(\bar{e}^n-\hat{e}^n))-(1-\alpha)\Sigma_{n=1}^{r-1}\Delta t(f^n,(D_t+D_{\bar{t}})(\bar{e}^n-\hat{e}^n))\\
&+\beta\Delta t^2\Sigma_{n=1}^{r-1}\Delta t(D_tD_{\bar{t}}f^n,D_t\hat{e}^n+D_{\bar{t}}\hat{e}^n)+\Sigma_{n=1}^{r-1}\Delta t(\Delta tw^n,D_t\hat{e}^n+D_{\bar{t}}\hat{e}^n).
\end{aligned}
\tag{13}
$$

Then it holds that

$$a(\hat{e}^0,\hat{e}^0)=0,\quad a(\hat{e}^0,\Delta tD_t\hat{e}^0)=0,$$

and

$$\beta\Delta t^2a(D_{\bar{t}}\hat{e}^r,D_{\bar{t}}\hat{e}^r)\leq\beta\Delta t^2A\{\alpha\|D_{\bar{t}}\hat{e}^r\|^2+(1-\alpha)\|D_{\bar{t}}\bar{e}^r\|^2\},$$

$$\beta\Delta t^2a(D_t\hat{e}^0,D_t\hat{e}^0)\leq\beta\Delta t^2A\{\alpha\|D_t\hat{e}^0\|^2+(1-\alpha)\|D_t\bar{e}^0\|^2\}$$

from Lemma 1, and

$$a(\hat{e}^r,\Delta tD_{\bar{t}}\hat{e}^r)\leq\frac{\varepsilon}{2}a(\hat{e}^r,\hat{e}^r)+\frac{A\Delta t^2}{2\varepsilon}\{\alpha\|D_{\bar{t}}\hat{e}^r\|^2+(1-\alpha)\|D_{\bar{t}}\bar{e}^r\|^2\}$$

for an arbitrary positive number ε. Applying (9) and (10), the eighth and ninth terms of the right hand side of (13) are estimated by

$$
\begin{aligned}
&\alpha\Sigma_{n=1}^{r-1}\Delta t(D_tD_{\bar{t}}(\hat{u}^n-u^n),D_t\hat{e}^n+D_{\bar{t}}\hat{e}^n)\\
&\quad\leq\alpha\Sigma_{n=1}^{r-1}\Delta t\|D_tD_{\bar{t}}(\hat{u}^n-u^n)\|^2+\alpha\Sigma_{n=1}^{r}\Delta t\|D_{\bar{t}}\hat{e}^n\|^2\\
&\quad\leq C_1h^2+\alpha\Sigma_{n=1}^{r}\Delta t\|D_{\bar{t}}\hat{e}^n\|^2,
\end{aligned}
$$

and

$$(1-\alpha)\Sigma_{n=1}^{r-1}\Delta t(D_tD_{\bar t}(\bar u^n-u^n),D_t\hat e^n+D_{\bar t}\bar e^n)$$

$$\leq C_2h^2+(1-\alpha)\Sigma_{n=1}^{r}\Delta t\|D_{\bar t}\bar e^n\|^2.$$

The tenth term of the right hand side of (13) is estimated by

$$\Sigma_{n=1}^{r-1}\Delta ta(\hat u^n-u^n,D_t\hat e^n+D_{\bar t}\hat e^n)=-\Sigma_{n=1}^{r-1}\Delta ta((D_t+D_{\bar t})(\hat u^n-u^n),\hat e^n)$$

$$+a(\hat u^r-u^r,\hat e^{r-1})+a(\hat u^{r-1}-u^{r-1},\hat e^r)-a(\hat u^0-u^0,\hat e^1)-a(\hat u^1-u^1,\hat e^0)$$

$$=-\Sigma_{n=1}^{r-1}\Delta ta((D_t+D_{\bar t})(\hat u^n-u^n),\hat e^n)+a(\hat u^r-u^r,\hat e^r)-a(\hat u^r-u^r,\Delta tD_{\bar t}\hat e^r)+$$
$$a(\hat u^{r-1}-u^{r-1},\hat e^r)-a(\hat u^0-u^0,\Delta tD_t\hat e^0)$$

$$\leq\frac{1}{2}\Sigma_{n=1}^{r-1}\Delta ta((D_t+D_{\bar t})(\hat u^n-u^n),(D_{\bar t}+D_t)(\hat u^n-u^n))+\frac{1}{2}\Sigma_{n=1}^{r-1}\Delta ta(\hat e^n,\hat e^n)+$$
$$\frac{1}{\delta}a(\hat u^r-u^r,\hat u^r-u^r)+\frac{1}{2\delta}a(\hat u^{r-1}-u^{r-1},\hat u^{r-1},u^{r-1})+\delta a(\hat e^r,\hat e^r)+$$
$$\frac{\delta\Delta t^2}{2}a(D_{\bar t}\hat e^r,D_{\bar t}\hat e^r)+\frac{1}{2\delta}a(\hat u^0-u^0,\hat u^0-u^0)+\frac{\delta\Delta t^2}{2}a(D_t\hat e^0,D_t\hat e^0)$$

$$\leq C_3h^2+\frac{1}{2}\Sigma_{n=1}^{r-1}\Delta ta(\hat e^n,\hat e^n)+\delta a(\hat e^r,\hat e^r)+\frac{\delta\Delta t^2}{2}A\{\alpha\|D_{\bar t}\hat e^r\|^2+(1-\alpha)\|D_{\bar t}\bar e^r\|^2\}$$
$$+\frac{\delta\Delta t^2}{2}A\{\alpha\|D_t\hat e^0\|^2+(1-\alpha)\|D_t\bar e^0\|^2\}$$

for an arbitrary positive number δ. The eleventh term of the right hand side of (13) is estimated by

$$\beta\Delta t^2\Sigma_{n=1}^{r-1}\Delta ta(D_tD_{\bar t}(\hat u^n-u^n),D_t\hat e^n+D_{\bar t}\hat e^n)$$

$$\leq\beta\Delta t^2\Sigma_{n=1}^{r-1}\Delta ta(D_tD_{\bar t}(\hat u^n-u^n),D_tD_{\bar t}(\hat u^n-u^n))+\beta\Delta t^2\Sigma_{n=1}^{r}\Delta ta(D_{\bar t}\hat e^n,D_{\bar t}\hat e^n)$$

$$\leq C_4h^2+\beta\Delta t^2A\Sigma_{n=1}^{r}\Delta t\{\alpha\|D_{\bar t}\hat e^n\|^2+(1-\alpha)\|D_{\bar t}\bar e^n\|^2\}.$$

Using Lemma 3, the twelveth and thirteenth terms of the right hand side of (13) are estimated by

$$(1-\alpha)\Sigma_{n=1}^{r-1}\Delta t(D_tD_{\bar t}u^n,(D_t+D_{\bar t})(\bar e^n-\hat e^n))$$

$$= -\Sigma_{n=1}^{r-1}\Delta t((D_t+D_{\bar t})D_tD_{\bar t}u^n,\bar e^n-\hat e^n)+(D_tD_{\bar t}u^r,\bar e^r-\hat e^r)-(D_tD_{\bar t}u^r,\Delta tD_{\bar t}(\bar e^r-\hat e^r))$$

$$+(D_tD_{\bar t}u^{r-1},\bar e^r-\hat e^r)-(D_tD_{\bar t}u^0,\bar e^0-\hat e^0)-(D_tD_{\bar t}u^1,\bar e^0-\hat e^0)-$$

$$(D_tD_{\bar t}u^0,\Delta tD_t(\bar e^0-\hat e^0))$$

$$\leq \Sigma_{n=1}^{r-1}\Delta tC_5h\sqrt{a(\hat e^n,\hat e^n)}+C_6h\sqrt{a(\hat e^r,\hat e^r)}+C_7\Delta th\sqrt{a(D_{\bar t}\hat e^r,D_{\bar t}\hat e^r)}+$$

$$C_8h\sqrt{a(\hat e^r,\hat e^r)}+C_9\Delta th\sqrt{a(D_t\hat e^0,D_t\hat e^0)}$$

$$\leq \Sigma_{n=1}^{r-1}\Delta t\{\frac{C_5^2h^2}{2\delta}+\frac{\delta}{2}a(\hat e^n,\hat e^n)\}+\{\frac{C_6^2h^2}{2\delta}+\frac{\delta}{2}a(\hat e^r,\hat e^r)\}+$$

$$\{\frac{C_7^2h^2}{2\delta}+\frac{\delta\Delta t^2}{2}a(D_{\bar t}\hat e^r,D_{\bar t}\hat e^r)\}+\{\frac{C_8^2h^2}{2\delta}+\frac{\delta}{2}a(\hat e^r,\hat e^r)\}+$$

$$\{\frac{C_9^2h^2}{2\delta}+\frac{\delta\Delta t^2}{2}a(D_t\hat e^0,D_t\hat e^0)\}$$

$$\leq C_{10}h^2+\delta a(\hat e^r,\hat e^r)+\frac{\delta}{2}\Sigma_{n=1}^{r-1}\Delta ta(\hat e^n,\hat e^n)+\frac{\delta\Delta t^2}{2}A\{\alpha\|D_{\bar t}\hat e^r\|^2+(1-\alpha)\|D_{\bar t}\bar e^r\|^2\}$$

$$+\frac{\delta\Delta t^2}{2}A\{\alpha\|D_t\hat e^0\|^2+(1-\alpha)\|D_t\bar e^0\|^2\},$$

and

$$(1-\alpha)\Sigma_{n=1}^{r-1}\Delta t(f^n,(D_t+D_{\bar t})(\bar e^n-\hat e^n))$$

$$\leq C_{11}h^2+\delta a(\hat e^r,\hat e^r)+\frac{\delta}{2}\Sigma_{n=1}^{r-1}\Delta ta(\hat e^n,\hat e^n)+\frac{\delta\Delta t^2}{2}A\{\alpha\|D_{\bar t}\hat e^r\|^2+(1-\alpha)\|D_{\bar t}\bar e^r\|^2\}$$

$$+\frac{\delta\Delta t^2}{2}A\{\alpha\|D_t\hat e^0\|^2+(1-\alpha)\|D_t\bar e^0\|^2\}$$

for an arbitrary positive number δ. The fourteenth and fifteenth terms of the right hand side of (13) are estimated by

$$\Sigma_{n=1}^{r-1}\Delta t(\Delta tw^n,D_t\hat e^n+D_{\bar t}\hat e^n)\leq \Delta t^2\Sigma_{n=1}^{r-1}\Delta t\|w^n\|^2+\Sigma_{n=1}^{r}\Delta t\|D_{\bar t}\hat e^n\|^2$$

$$\leq\Delta t^2\Sigma_{n=1}^{r-1}\Delta t\|w^n\|^2+\frac{1}{\alpha}\Sigma_{n=1}^{r}\Delta t\{\alpha\|D_{\bar t}\hat e^n\|^2+(1-\alpha)\|D_{\bar t}\bar e^n\|^2\}$$

$$\leq C_{12}\Delta t^2+\frac{1}{\alpha}\Sigma_{n=1}^{r}\Delta t\{\alpha\|D_{\bar t}\hat e^n\|^2+(1-\alpha)\|D_{\bar t}\bar e^n\|^2\},$$

and

$$\beta\Delta t^2\Sigma_{n=1}^{r-1}\Delta t(D_t D_{\bar{t}} f^n, D_t\hat{e}^n + D_{\bar{t}}\hat{e}^n) \leq C_{13}\Delta t^2 + \frac{\beta}{\alpha}\Sigma_{n=1}^{r}\Delta t\{\alpha\|D_{\bar{t}}\hat{e}^n\|^2 + (1-\alpha)\|D_{\bar{t}}\bar{e}^n\|^2\}.$$

Therefore, summing up these estimates, we can obtain

$$\begin{aligned}
&\alpha\|D_{\bar{t}}\hat{e}^r\|^2 + (1-\alpha)\|D_{\bar{t}}\bar{e}^r\|^2 + a(\hat{e}^r,\hat{e}^r) \\
&\quad \leq \alpha\|D_t\hat{e}^0\|^2 + (1-\alpha)\|D_t\bar{e}^0\|^2 + C_{14}(h^2+\Delta t) + (\varepsilon/2 + 3\delta)a(\hat{e}^r,\hat{e}^r) + \\
&\quad\quad \max\{0, \frac{1}{2\varepsilon} + \frac{3\delta}{2} - \beta\}\Delta t^2 A\{\alpha\|D_{\bar{t}}\hat{e}^r\|^2 + (1-\alpha)\|D_{\bar{t}}\bar{e}^r\|^2\} + \\
&\quad\quad C_{15}\Sigma_{n=1}^{r}\Delta t\{\alpha\|D_{\bar{t}}\hat{e}^n\|^2 + (1-\alpha)\|D_{\bar{t}}\bar{e}^n\|^2 + a(\hat{e}^n,\hat{e}^n)\},
\end{aligned}$$

that is,

$$\begin{aligned}
&[\,1 - \frac{\Delta t^2 A_m(m+1)(m+2)}{\kappa^2\{m+2-(m+1)\alpha\}}\cdot\max\{0, \frac{1}{2\varepsilon} + \frac{3\delta}{2} - \beta\}\,]\{\alpha\|D_{\bar{t}}\hat{e}^r\|^2 + (1-\alpha)\|D_{\bar{t}}\bar{e}^r\|^2\} \\
&\quad + (1-\varepsilon/2 - 3\delta)a(\hat{e}^r,\hat{e}^r) \\
&\leq C_{16}(h^2+\Delta t^2) + C_{17}\Sigma_{n=1}^{r}\Delta t\{\alpha\|D_{\bar{t}}\hat{e}^n\|^2 + (1-\alpha)\|D_{\bar{t}}\bar{e}^n\|^2 + a(\hat{e}^n,\hat{e}^n)\}
\end{aligned}$$

for sufficiently small $\delta > 0$. Then, from the stability condition, we have the following inequality

$$\begin{aligned}
&\alpha\|D_{\bar{t}}\hat{e}^r\|^2 + (1-\alpha)\|D_{\bar{t}}\bar{e}^r\|^2 + a(\hat{e}^r,\hat{e}^r) \\
&\qquad \leq C_{18}(h^2+\Delta t^2) + C_{19}\Sigma_{n=1}^{r}\Delta t\{\alpha\|D_{\bar{t}}\hat{e}^n\|^2 + (1-\alpha)\|D_{\bar{t}}\bar{e}^n\|^2 + a(\hat{e}^n,\hat{e}^n)\}.
\end{aligned}$$

Applying Lemma 2 yields

$$\begin{aligned}
\alpha\|D_{\bar{t}}\hat{e}^r\|^2 + (1-\alpha)\|D_{\bar{t}}\bar{e}^r\|^2 + a(\hat{e}^r,\hat{e}^r) &\leq C_{18}(h^2+\Delta t^2)\{1/(1-C_{19}\Delta t)^r - 1\} \\
&\leq C_{18}(h^2+\Delta t^2)\{1/(1-C_{19}\Delta t)^{T/\Delta t} - 1\}
\end{aligned}$$

for sufficiently small Δt. By the fact that $1/(1-C_{19}\Delta t)^{T/\Delta t} \longrightarrow e^{C_{19}T}$ as $\Delta t \longrightarrow 0$, we have

$$\alpha\|D_{\bar{t}}\hat{e}^r\|^2+(1-\alpha)\|D_{\bar{t}}\bar{e}^r\|^2+a(\hat{e}^r,\hat{e}^r) \leq C_{20}(h^2+\Delta t^2).$$

From Lemma 4, we can obtain the desired inequality

$$\alpha\|\hat{e}^r\|^2+(1-\alpha)\|\bar{e}^r\|^2+\alpha\|D_{\bar{t}}\hat{e}^r\|^2+(1-\alpha)\|D_{\bar{t}}\bar{e}^r\|^2+a(\hat{e}^r,\hat{e}^r) \leq \bar{C}(h^2+\Delta t^2)$$

where $\bar{C}$ is a positive constant independent of h and Δt. This completes the proof.

Theorem 3. Let $\{\hat{v}^r,\bar{v}^r\}$ be the solutions of (6). If the stability condition is satisfied, then, for sufficiently small Δt, there exists a constant $\tilde{C}$ which is independent of h and Δt, such that

$$\alpha\|\hat{E}^r\|^2+(1-\alpha)\|\bar{E}^r\|^2+\alpha\|D_{\bar{t}}\hat{E}^r\|^2+(1-\alpha)\|D_{\bar{t}}\bar{E}^r\|^2+\Sigma_{i=1}^m\|\partial\hat{E}^r/\partial x_i\|^2$$
$$\leq \tilde{C}(h^2+\Delta t^2), \qquad r=2,3,\cdots,p,$$

where $\hat{E}^r=u^r-\hat{v}^r$, $\bar{E}^r=u^r-\bar{v}^r$.

Proof. Define a space $L_2(\Omega)\times L_2(\Omega)$, each element of which is a pair of functions $\{u_1,u_2\}(u_1,u_2\in L_2(\Omega))$. Addition and scalar multiplication are defined in the obvious manner. The inner product and the norm on $L_2(\Omega)\times L_2(\Omega)$ are defined by

$$[\{u,v\},\{w,z\}] = \alpha(u,w)+(1-\alpha)(v,z),$$

$$|||\{u,v\}||| = [\{u,v\},\{u,v\}]^{1/2}.$$

Using the triangle inequality, for sufficiently small Δt, there exists a constant $\tilde{C}$, which is independent of h and Δt, such that

$$|||\{\hat{E}^r,\bar{E}^r\}|||^2+|||\{D_{\bar{t}}\hat{E}^r,D_{\bar{t}}\bar{E}^r\}|||^2+a(\hat{E}^r,\hat{E}^r)$$
$$\leq 2|||\{u^r-\hat{u}^r,u^r-\bar{u}^r\}|||^2+2|||\{\hat{e}^r,\bar{e}^r\}|||^2+2|||\{D_{\bar{t}}(u^r-\hat{u}^r),D_{\bar{t}}(u^r-\bar{u}^r)\}|||^2+$$
$$2|||\{D_{\bar{t}}\hat{e}^r,D_{\bar{t}}\bar{e}^r\}|||^2+2a(u^r-\hat{u}^r,u^r-\hat{u}^r)+2a(\hat{e}^r,\hat{e}^r) \leq \tilde{C}(h^2+\Delta t^2)$$

from Theorem 2 and (9),(10). This completes the proof.

4. Numerical Experiments

To illustrate the efficiency of our scheme, some numerical results are obtained for the two dimensional problem($m=2$). Let Ω be a unit square domain defined by

$$\Omega : \quad 0 < x < 1, \quad 0 < y < 1.$$

Example.

$$\partial^2 u/\partial t^2 = \Delta u \qquad \text{in } \Omega, \quad 0 < t \leq \sqrt{2}/2,$$

$$u(x,y,t)=0 \qquad \text{on } \Gamma,$$

$$u(x,y,0)=0 \qquad (x,y) \in \Omega,$$

$$\frac{\partial}{\partial t} u(x,y,0)=100\sqrt{2}\pi \sin(\pi x)\cdot\sin(\pi y) \qquad (x,y) \in \Omega.$$

The exact solution is given by

$$u(x,y,t)=100\sin(\pi x)\cdot\sin(\pi y)\cdot\sin(\sqrt{2}\pi t).$$

The square domain is divided into uniform mesh with isosceles triangles(9,25 and 81 nodes). We also divide the time interval into 6,12 and 24 equal parts, each of which corresponding to the above mesh nodes. The computations were performed for the parameters $\alpha=0,1/2,1$, and $\beta=0,1/4,1$. All the cases satisfy the stability condition of our theorem(see Table 1).

Table 2 and Figure 1 show the results for the value of the center of the square domain $\hat{v}(1/2,1/2,t)$, compared with the exact value $u(1/2,1/2,t)(t=\sqrt{2}/6, \sqrt{2}/4, \sqrt{2}/3, 5\sqrt{2}/12, \sqrt{2}/2)$. We can see that the GMM solutions converge to the exact values with h and Δt. In particular, the case of $\alpha=1/2$ shows better agreements with the exact values than the other cases of $\alpha=0$ and $\alpha=1$.

All the computations are performed by the single precision arithmetic on FACOM 230-28 computer in Ehime University.

Table 1. Mesh ratio

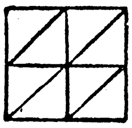
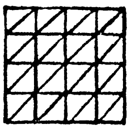
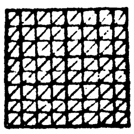
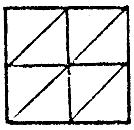
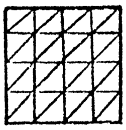
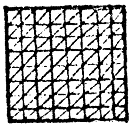
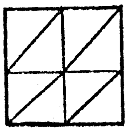
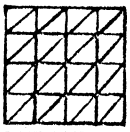
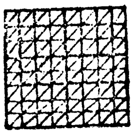
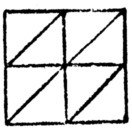
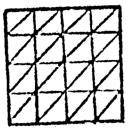
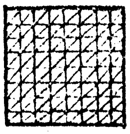
mesh (node)		σ	h	κ	Δt	$\sqrt{1-4\beta}\cdot\Delta t/\kappa$	$\sqrt{\frac{4\{m+2-(m+1)\alpha\}}{(m+1)(m+2)A_m}}$
9		0	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{4}$	$\frac{\sqrt{2}}{12}$	$\sqrt{(1-4\beta)/9}$	$\sqrt{(4-3\alpha)/6}$
25		0	$\frac{\sqrt{2}}{4}$	$\frac{\sqrt{2}}{8}$	$\frac{\sqrt{2}}{24}$	$\sqrt{(1-4\beta)/9}$	$\sqrt{(4-3\alpha)/6}$
81		0	$\frac{\sqrt{2}}{8}$	$\frac{\sqrt{2}}{16}$	$\frac{\sqrt{2}}{48}$	$\sqrt{(1-4\beta)/9}$	$\sqrt{(4-3\alpha)/6}$

Table 2. The results for Example.

β	mesh	α	t: $\sqrt{2}/6$	$\sqrt{2}/4$	$\sqrt{2}/3$	$5\sqrt{2}/12$	$\sqrt{2}/2$
1		0	95.20	120.73	124.31	105.29	67.12
		0.5	92.75	111.94	105.55	75.03	27.36
		1	88.61	97.59	76.55	31.95	-22.48
		0	89.30	106.35	97.80	65.70	17.80
		0.5	87.91	102.10	89.52	53.46	3.35
		1	86.13	97.02	80.14	40.18	-11.71
		0	87.32	101.68	89.53	54.05	4.48
		0.5	86.90	100.45	87.17	50.63	0.56
		1	86.46	99.14	84.72	47.16	-3.40
$\frac{1}{4}$		0	93.70	115.31	112.64	86.26	41.72
		0.5	90.28	103.29	87.81	48.10	-4.87
		1	83.78	81.68	46.91	-6.62	-57.50
		0	88.59	104.17	93.48	59.24	10.06
		0.5	87.05	99.54	84.58	46.29	-4.83
		1	85.03	94.07	74.65	32.10	-20.68
		0	87.11	101.06	88.34	52.31	2.48
		0.5	86.69	99.80	85.94	48.84	-1.47
		1	86.23	98.46	83.46	45.33	-5.48
0		0	93.08	113.12	108.02	78.92	32.28
		0.5	89.21	99.62	80.52	37.56	-16.53
		1	81.45	74.34	34.19	-21.16	-67.10
		0	88.34	103.40	91.97	56.99	7.42
		0.5	86.74	98.63	82.85	43.80	-7.61
		1	84.63	93.07	72.69	29.19	-23.69
		0	87.04	100.85	87.94	51.72	1.80
		0.5	86.61	99.59	85.52	48.24	-2.15
		1	86.15	98.22	83.04	44.69	-6.17
exact			86.60	100.00	86.60	50.00	0.00

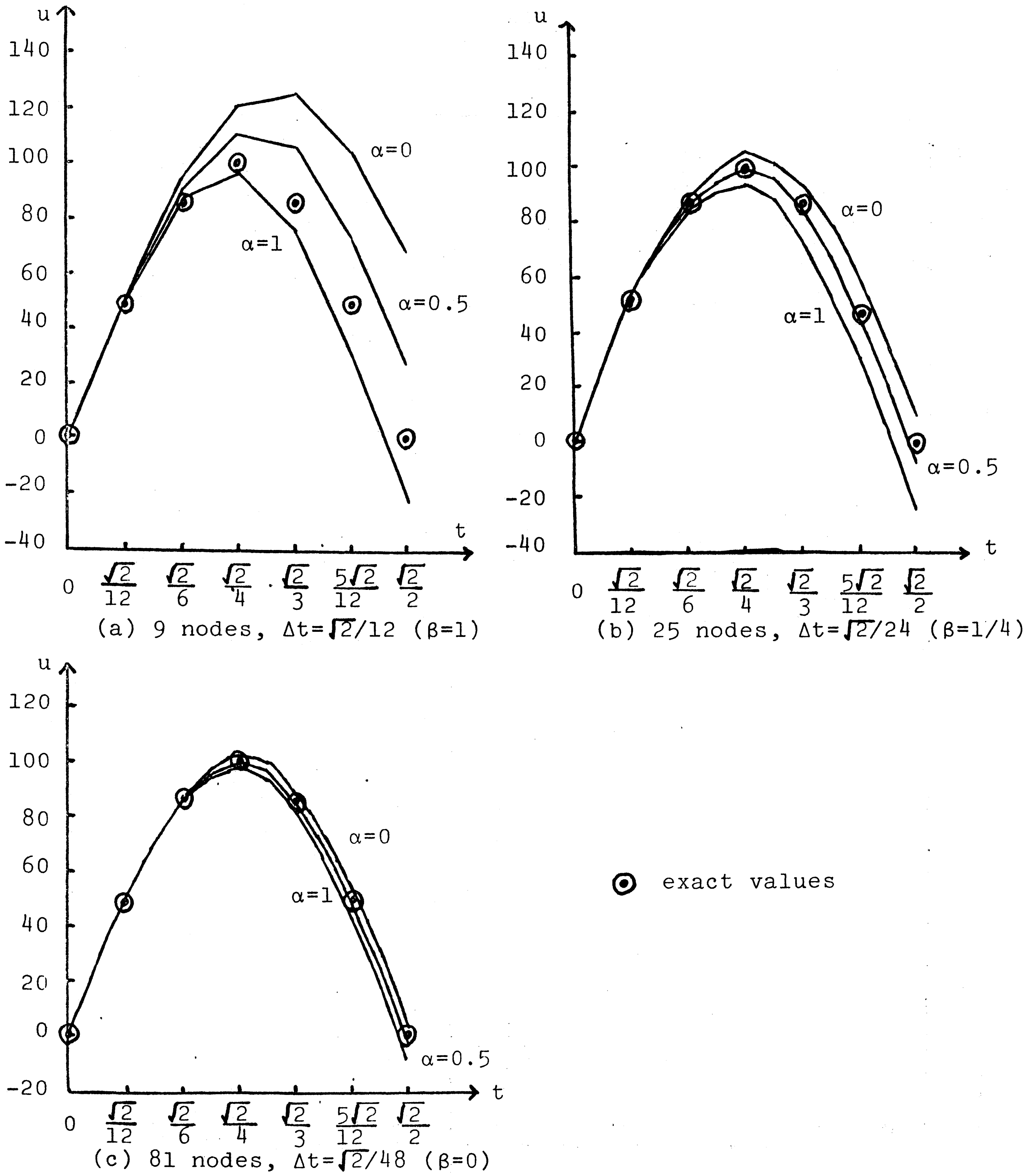


Figure 1. Convergence for the values of the center

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On the explicit finite difference approximation of the Navier-Stokes equation in a non cylindrical domain

By Masa-Aki NAKAMURA

Introduction

This paper concerns the numerical method of the Navier-Stokes equation in a region with boundaries which may vary as the time t varies. We restrict the case of 2-dimensional space variable. H. Fujita and N. Sauer established the existence and the uniqueness of the weak solution of this problem by the penalty method in [3]. We adopt this method to treat the moving boundaries. R. Temam introduced a method to approximate the Navier-Stokes equation with the equation of Cauchy-Kowalevskaja type in [5]. His method has the practical importance to treat the nonlinear term $u \cdot \nabla u$ and the condition div u = 0. So we use a discrete version of this approximation method also.

The most significant feature of our finite difference scheme is in its pure expliciteness. Namely we can get the numerical solution by step by step integration in time without the inversion of any matrix.

In §1 the result for the continuous problem will be summarized after preparing some notations and terminologies. We will describe our scheme in §2. The stability of this scheme will be investigated in §3. And finally the convergence of the approximate solution will be established in §4.

§1. A summary of the continuous problem

The scalar product and the norm are denoted by $(\cdot,\cdot)$ and $|\cdot|$ on $L^2(G)$ (resp. $((\cdot,\cdot))$ and $\|\cdot\|$ on $H_0^1(G)$), where the set G is a bounded open domain in R^2 with a smooth boundary. When it is necessary to distinguish the set G ,they will be written as $(\cdot,\cdot)_G$, $|\cdot|_G$,$((\cdot,\cdot))_G$ and $\|\cdot\|_G$. Frequently the direct product spaces of m-copies of $L^2(G)$ and $H_0^1(G)$ are considered,which are also denoted by $L^2(G)$ and $H_0^1(G)$.
For m = 2 ,the norm of $H_0^1(G)$ is taken as

$$\| u \| = (\nabla u, \nabla u) = \sum_{i,j=1}^{2} \left(\frac{\partial u_j}{\partial x_i} , \frac{\partial u_j}{\partial x_i} \right)$$

$$\text{for} \quad u = (u_j)_{j=1,2} \in H_0^1(G) .$$

The following notations are also used .

$$V(G) = \{ u \in H_0^1(G) \; ; \; \operatorname{div} u = 0 \} ,$$

$$H(G) = L^2\text{- completion of } \{ u \in C_0^\infty(G) \; ; \; \operatorname{div} u = 0 \} .$$

Let T be a positive finite number. Consider a family $\Omega(t)$, $0 \leq t \leq T$, of simply connected bounded open domains in R^2 . The boundaries, $\Gamma(t) = \partial\Omega(t)$, are assumed to be smooth. Let us write

$$\hat{\Omega} = \bigcup_{0 \leq t \leq T} [\{t\} \times \Omega(t)] ,$$

$$\hat{\Gamma} = \bigcup_{0 \leq t \leq T} [\{t\} \times \partial\Omega(t)] .$$

(Assumption)

i) As t varies, $\Gamma(t)$ changes smoothly in the sense that the (t,x)-surface $\hat{\Gamma}$ is covered by a finite number of patches and in each patch, $\hat{\Gamma}$ can be represented by $x_1' = \phi(t,x_2')$ in terms of a C^3-class function ϕ of 2-variables under a suitable choice of coordinates (x_1', x_2') in R^2.

ii) There exists a bounded open domain B in R^2 such that the boundary ∂B is smooth, $\Omega(t) \subset B$ for all t, and $\mathrm{dist}(\partial B, \Gamma(t)) \geqq \delta_0 > 0$ for all t.

Our continuous problem is the following initial boundary value problem :

$$\begin{cases} \dfrac{\partial u}{\partial t} - \nu\Delta u + u\cdot\nabla u + \nabla p = f(t,x) & \text{in } \hat{\Omega}, \\ \mathrm{div}\, u = 0 & \text{in } \hat{\Omega}, \\ u = 0 & \text{on } \hat{\Gamma}, \\ u(0,x) = u_0(x) & \text{in } \hat{\Omega}(0). \end{cases}$$

where $u = (u_1(t,x), u_2(t,x))$ is the flow velocity and $p = p(t,x)$ is the pressure, and ν is a positive constant.

Consider the weak formulation of this problem.

Problem 1. For given functions $u_0 \varepsilon H(\Omega(0))$ and $f \varepsilon L^2(0,T;H(\Omega(t)))$, find $u \varepsilon L^2(0,T;V(\Omega(t))) \cap L^\infty(0,T;H(\Omega(t)))$ satisfying

$$\int_0^T \{-(u,\phi_t) + \nu((u,\phi)) + b(u,u,\phi)_{\Omega(t)}\}dt$$
$$= \int_0^T (f,\phi)dt + (u_0,\phi(0)) \quad \text{for any } \phi \varepsilon \hat{D}_\sigma(\hat{\Omega}).$$

In the above problem, $f \varepsilon L^2(0,T;H(\Omega(t)))$ implies that $f(t,x) \varepsilon H(\Omega(t))$ for almost every $t \varepsilon [0,T]$ satisfying

$$\int_0^T |f(t,\cdot)|^2_{\Omega(t)}\, dt < \infty .$$

The spaces $L^2(0,T;V(\Omega(t)))$ and $L^\infty(0,T;H(\Omega(t)))$ are defined analogously. The trilinear form $b(u,v,w)_G$ is defined as follows

$$b(u,v,w) = \frac{1}{2}\sum_{i,j=1}^{2}\int_G (u_i\frac{\partial v_j}{\partial x_i}w_j - u_i v_j \frac{\partial w_j}{\partial x_i})dx \quad .$$

We use the abbreviation $\phi(t)$ for the function $\phi(t,x)$ when it is considered as an element of some function space in x-variables. Finally

$$\hat{D}_\sigma(\hat{\Omega}) = \{\phi \in C^\infty(\hat{\Omega}) \ ; \ \mathrm{div}\ \phi = 0,\ \mathrm{supp}\ \phi \subset \hat{\Omega}, \phi(T) = 0 \ \}.$$

By some standard calculation, we can conclude that the smooth solution of the original problem is the solution of Problem 1 . Note that

$$b(u,u,v) = (u\cdot\nabla u,\ v) \quad \text{if div } u = 0.$$

It is also remarked that

$$b(u,v,v) = 0 \quad \text{for} \quad u,v \in H_0^1(G) \quad .$$

Theorem 1 . (Fujita - Sauer [3])

Under (Assumption), there exists a unique solution of Problem 1 .

§2. The explicit finite difference scheme

The mesh size of space variables and of time variable are denoted by h and k respectively:

$$h = \Delta x_1 = \Delta x_2 \quad , \quad k = \Delta t \quad .$$

Hereafter we denote by B , the set in (Assumption) ii) . We prepare some notations and symbols.

$R_h = \{ M \in R \ ; \ M = (m_1 h\ ,\ m_2 h),\ m_i \in Z \ \}$;

$\tau_h(M,0) = \tau_h(m) = ((m_1-\frac{1}{2})h,(m_1+\frac{1}{2})h)\times((m_2-\frac{1}{2})h,(m_2+\frac{1}{2})h)$;

$\tau_h(M,1) = \bigcup_{i=0,1, j=0,1} \tau_h(M + (ih,jh))$;

$w_{hM}(x)$; the characteristic funtion of $\tau_h(M)$;

∇_i, $\bar{\nabla}_i$, $i = 1,2$, the forward and the backward difference operators:

$$\nabla_i \phi(x) = \frac{\phi(x+\vec{h}_i) - \phi(x)}{h}, \quad \overline{\nabla}_i \phi(x) = \frac{\phi(x) - (x-\vec{h}_i)}{h},$$

where $\vec{h}_i = (\delta_{ij} h)$, $j = 1,2$.

An open set B is approximated by the set B_h ;

$$B_h = \cup \{ x \varepsilon \tau_h(M) \;;\; M \varepsilon R_h, \; \tau_h(M,1) \subset B \} .$$

Consider the function space ;

$$V_h(B_h) = \{ u_h(x) = \sum_{M \varepsilon B_h \cap R_h} u_h(M)\, w_{hM}(x) \;;\; u_h(M) \varepsilon R^2 \} .$$

The operator ∇_i is regarded as an operator in the space $V_h(B_h)$ by the formula :

$$\nabla_i u_h(x) = \sum_{M \varepsilon B_h \cap R_h} (\nabla_i u_h)(M)\, w_{hM}(x) .$$

Analogously we define the operator $\overline{\nabla}_i$.

The functions u_h and $\nabla_i u_h$,i=1,2, have compact supports in B , by the definition of V_h and B_h. Hence they will be considered as functions defined on R^2 .

The following scalar products and norms are introduced on the space V_h :

$$(u_h, v_h)_h = \int_B u_h(x) v_h(x)\, dx , \quad |u_h|_h = (u_h, u_h)_h^{\frac{1}{2}} ,$$

$$((u_h, v_h))_h = \sum_{i=1}^{2} \int_B (\nabla_i u_h(x))(\nabla_i v_h(x)) dx ,$$

$$\| u_h \|_h = ((u_h, u_h))_h^{\frac{1}{2}} .$$

The suffix h of these scalar products and norms will be omitted.

Proposition 1 (Discrete Poincaré inequality and its inverse) For any $u_h \varepsilon V_h(B_h)$, we have

(1) $|u_h| \leqq C_0 \|u_h\|$, C_0= diameter of B ,

(2) $\|u_h\| \leqq S(h)|u_h|$, $S(h) = 2\sqrt{2}/h$.

Define the bilinear mapping g_h from $V_h \times V_h$ to V_h , and the trilinear form b_h on $V_h \times V_h \times V_h$ by the following formulas :

(3) $g_h(v_h,u_h)_j = \frac{1}{2}\sum_{i=1}^{2} \{ v_{ih} \nabla_i u_{jh} + (\bar{\nabla}_i v_{ih})u_{jh} + \bar{v_{ih}}\bar{\nabla}_i u_{jh} \}$,

where $\bar{v_{ih}}(x) = v_{ih}(x - \vec{h}_i)$,

(4) $b_h(u_h,v_h,w_h) = (g_h(u_h,v_h),w_h)$.

Then the following equalities and the estimate hold (see Temam [ ]).

(5) $b_h(u_h,v_h,w_h) = \frac{1}{2}\sum_{i,j=1}^{2} \int u_{ih}\{\nabla_i v_{jh}w_{jh} - v_{jh}\nabla_i w_{jh}\}dx$,

(6) $b_h(u_h, v_h, v_h) = 0$,

(7) $|b_h(u_h, v_h,w_h)| \leqq |u_h|^{\frac{1}{2}}\| u_h\|^{\frac{1}{2}}\{\| v_h\| \, |w_h|^{\frac{1}{2}}\|w_h\|^{\frac{1}{2}} + |v_h|^{\frac{1}{2}}\| v_h\|^{\frac{1}{2}}\| w_h\| \}$.

Define the restriction operator ρ_h from $L^2(B)$ to $V_h(B_h)$ as follows,

$$\rho_h u = u_h , \quad u_h(M) = \frac{1}{h^2}\int_{\tau_h(M)} u(x)dx, \; M\epsilon R_h \cap B_h .$$

The functions u_{0h} and $p_{0h}\epsilon V_h(\Omega(0)_h)$ are extended to the functions $\bar{u}_h^0$ and $\bar{p}_h^0 \epsilon V_h(B_h)$ which vanish outside $\Omega(0)_h$. For a positive integer N , and $k = T/N$, we put ,

$$f_h^n = \frac{1}{k}\int_{(n-1)k}^{nk} (\rho_h f)(s)ds , \quad n = 1,2, \ldots N$$

for $f \,\epsilon\, L^2(0,T;L^2(B))$.

Our scheme is the following :

(8) $u_h^0 = \bar{u}_h^0$,

$p_h^0 = \bar{p}_h^0$,

If $u_h^0, \dots, u_h^m$ and $p_h^0, \dots, p_h^m$ are determined, then define u_h^{m+1} and p_h^{m+1} by the formula :

$$(9) \quad \frac{1}{k}\{ u_h^{m+1} - u_h^m \} - \nu \sum_{i=1}^{2} \bar{\nabla}_i \nabla_i u_h^m + g_h(u_h^m, u_h^m) + \bar{\nabla} p_h^m + n\chi_h^m u_h^m = f_h^{m+1} ,$$

$$(10) \quad \frac{1}{k}\{ \varepsilon p_h^{m+1} - \varepsilon p_h^m \} + \sum_{i=1}^{2} \nabla_i u_h^m = 0 ,$$

where $\chi_h^m(M) = \begin{cases} 1 & \text{if } (mk, M) \in \hat{\Omega} ,\ M \in B_h \cap R_h \\ 0 & \text{otherwise} \end{cases}$,

and n is a positive integer .

This scheme is a discrete version of the following system.

$$\begin{cases} \frac{\partial u}{\partial t} - \nu\Delta u + u\cdot\nabla u + \frac{1}{2}(\operatorname{div} u)u + n\chi(t,x)u + \nabla p = f(t,x) & \text{in } \hat{B} , \\ \operatorname{div} u + \varepsilon\frac{\partial p}{\partial t} = 0 & \text{in } \hat{B} , \\ u = 0 & \text{on } \partial B , \\ u(0) = \bar{u}_0 & \text{in } B , \\ p(0) = \bar{p}_0 & \text{in } B . \end{cases}$$

In the above system , $\hat{B} = [0,T] \times B$, and $\chi(t,x)$ is the characteristic function of $\hat{B} - \hat{\Omega}$. The functions $\bar{u}_0$ and $\bar{p}_0$ are the natural extension of u_0 and p_0 which vanish on $B - \Omega(0)$.

This system was introduced by Temam [5] to the fixed boundary problem. To the moving boundary problem, we can show the existence and the uniqueness of the weak solution, and the convergence to the solution of Problem 1.

§3. The stability of the scheme.

Lemma 1. Let K and δ be arbitrary fixed positive numbers, and let $N = T/k$. Define the quantity L_ℓ for the solution u^ℓ of our scheme by the formula :

(11) $$L_\ell = \nu - 5kS(h)^2\{\nu^2 + 2|u^\ell|^2\} - \frac{2k^2}{\varepsilon} ,$$

$$\ell = 0,1,2,\ldots,N .$$

If the following conditions (12), (13) and (14) are satisfied,

(12) $0 < \delta \leqq L_\ell$, $\ell = 0,1,\ldots,m$,

(13) $10kS(h) \leqq K\varepsilon$,

(14) $0 < \delta \leqq 2 - 5kn$,

then we have

(15) $|u^{m+1}| + \varepsilon|p_h^{m+1}| \leqq C_1$,

(16) $k\sum_{\ell=0}^{m} \|u^\ell\|^2 \leqq C_2$,

(17) $k\sum_{\ell=0}^{m} n|\chi^\ell u^\ell|^2 \leqq C_2$,

where C_1 and C_2 are constants independent of ε,k,n and h .

(Proof) Multiplying (9) by $2u^m$, and (10) by $2p^m$, and integrating on the set B , we get the following two equalities.

$$|u^{m+1}|^2 - |u^m|^2 - \varepsilon|u^{m+1} - u^m|^2 + 2k\nu\|u^m\|^2 + 2kn|\chi^m u^m|^2 + 2k(\overline{\nabla}p^m,u^m) = 2k(f^{m+1},u^m) ,$$

$$\varepsilon|p^{m+1}|^2 - \varepsilon|p^m|^2 - \varepsilon|p^{m+1} - p^m|^2 + 2k(\nabla\cdot u^m,p^m) = 0 .$$

Adding these two equalities , and using the relation ;

$$2k(\nabla\cdot u^m,p^m) + 2k(\overline{\nabla}p^m,u^m) = 0 \quad ,$$

we obtain

(18) $$|u^{m+1}|^2 + \varepsilon|p^{m+1}|^2 - |u^m|^2 - \varepsilon|p^m|^2 + 2k\nu\|u^m\|^2 + 2kn|\chi^m u^m|^2$$
$$= \varepsilon|p^{m+1} - p^m|^2 + |u^{m+1} - u^m|^2 + 2k(f^{m+1},u^m) \quad .$$

Now we estimate the three terms in the right-hand side.

From (10), it follows that

$$\begin{aligned} 2\varepsilon|p^{m+1} - p^m|^2 &= -2k(\nabla\cdot u^m,p^{m+1} - p^m) \\ &\leqq 2k|\nabla\cdot u^m|\,|p^{m+1} - p^m| \\ &\leqq 2\sqrt{2}k\|u^m\|\,|p^{m+1} - p^m| \\ &\leqq \frac{2k^2}{\varepsilon}\|u^m\|^2 + \varepsilon|p^{m+1} - p^m|^2 \quad . \end{aligned}$$

Therefore it holds,

(19) $$\varepsilon|p^{m+1} - p^m|^2 \leqq \frac{2k^2}{\varepsilon}\|u^m\|^2 \quad .$$

Taking the scalar product of (9) and $u^{m+1}-u^m$, we have

(20) $$\begin{aligned} 2|u^{m+1} - u^m| &= -2k\nu((u^m,u^{m+1} - u^m)) \\ &\quad +2k\, b(u^m,u^m,u^{m+1} - u^m) \\ &\quad -2kn(\chi^m u^m,u^{m+1} - u^m) \\ &\quad -2k(\nabla\cdot(u^{m+1} - u^m),p^m) \\ &\quad +2k(f^{m+1},u^{m+1} - u^m) \quad . \end{aligned}$$

Each term in the right-hand side of (20) is majorized as follows ,

$$\begin{aligned} |2k\nu((u^m,u^{m+1} - u^m))| &\leqq 2k\nu\|u^m\|\,\|u^{m+1} - u^m\| \\ &\leqq \frac{1}{5}|u^{m+1} - u^m|^2 + 5k^2\nu^2 S(h)^2\|u^m\|^2 \quad , \end{aligned}$$

$$\begin{aligned} &|2kb(u^m,u^m,u^{m+1} - u^m)| \\ &\leqq \frac{1}{5}|u^{m+1} - u^m|^2 + 10k^2 S(h)^2|u^m|^2\|u^m\|^2 \quad , \end{aligned}$$

(Note the inequalities (2),(7)),

$$|2k(\chi^m u^m, u^{m+1} - u^m)| \leqq \frac{1}{5}|u^{m+1} - u^m|^2 + 5k^2 n^2 |\chi^m u^m|^2 ,$$

$$|2k(f^m, u^{m+1} - u^m)| \leqq \frac{1}{5}|u^{m+1} - u^m|^2 + 5k^2 |f^{m+1}|^2 ,$$

$$|2k(\nabla\cdot(u^{m+1} - u^m), p^m)| \leqq \frac{1}{5}|u^{m+1} - u^m|^2 + 10k^2 S(h)^2 |p^m|^2 ,$$

Substituting these estimates into (20), we obtain the estimate (21).

(21) $$|u^{m+1} - u^m|^2 \leqq 5k^2 S(h)^2 \{\nu^2 + 2|u^m|^2\} \| u^m \| + 5k^2 n^2 |\chi^m u^m|^2 + 5k^2 |f^{m+1}|^2 + 10k^2 S(h)^2 |p^m|^2 .$$

By the Schwarz' inequality and the inequality (6), it holds that

(22) $$|2k(f^{m+1}, u^m)| \leqq k\nu \| u^m \|^2 + \frac{kC_0^2}{\nu} |f^{m+1}|^2 .$$

The inequality (18) and the estimates (19), (21) and (22) imply the estimate (23).

(23) $$U^{m+1} - U^m + (2 - 5kn)kn|\chi^m v^m|^2 + L_m k \| u^m \|^2 \leqq (5k + \frac{C_0^2}{\nu})k|f^{m+1}|^2 + 10k^2 S(h)^2 |p^m|^2 ,$$

where $U^m = |u^m|^2 + \varepsilon |p^m|^2$.

Adding the inequalities (23) for $m = o, 1, \ldots, \ell$, we obtain

$$U^{\ell+1} + (2 - 5kn)k \sum_{m=0}^{\ell} n|\chi^m u^m|^2 + k \sum_{m=0}^{\ell} L_m \| u^m \|^2 \leqq M_\ell + Kk \sum_{m=0}^{\ell} U^m ,$$

where $M_\ell = k(5k + \frac{C_0^2}{\nu}) \sum_{m=0}^{\ell} |f^{m+1}|^2 + |u^0|^2 + |p^0|^2$.

Hence it follows from the conditions (12), (13) and (14) that

(24) $$U^{\ell+1} + \delta k \sum_{m=0}^{\ell} n|\chi^m u^m|^2 + \delta k \sum_{m=0}^{\ell} \| u^m\|^2 \leqq M_\ell + Kk \sum_{m=0}^{\ell} U^m \quad .$$

Let

(25) $$M = (5T + \frac{C_0^2}{\nu}) \int_0^T |f(t)|^2 dt + |u^0|^2 + |p^0|^2$$

If $\varepsilon \leqq 1$, it holds

$$M_\ell \leqq M \ , \quad \ell = 1,\dots,N \quad .$$

Since $\delta > 0$, the inequality (24) implies

$$U^{\ell+1} \leqq M + Kk \sum_{m=0}^{\ell} U^m \quad .$$

Hence we obtain

$$U^{\ell+1} \leqq C_1 = Me^{kT} \quad .$$

This estimate and the inequality (24) imply the estimates (16) and (17). q.e.d.

Theorem 2. Consider the condition

(26) $$L = \nu - k[5S(h)^2\{\nu^2 + 2Me^{kT}\} + \frac{2k}{\varepsilon}] \geqq \delta > 0 \quad ,$$

where M is determined by (25).

If the conditions (13),(14) and (26) are satisfied, we have the following estimates with some constants C_1 and C_2 independent of ε, k, n and h .

(27) $$|u^\ell|^2 + \varepsilon|p^\ell|^2 \leqq C_1 \quad , \quad \ell = 0,1,\dots,N \quad ,$$

(28) $$k \sum_{\ell=0}^{N-1} \| u^\ell\|^2 \leqq C_2 \quad ,$$

(29) $$k \sum_{\ell=0}^{N-1} n|\chi^\ell u^\ell|^2 \leqq C_2 \quad .$$

(Proof) We can easily prove inductively that $L_\ell \geqq L \geqq \delta$.

q.e.d.

Now we introduce the linear operators $\overline{\omega} \in L(H_0^1(B), L^2(B))$,

$q_h \varepsilon L(V_h, L^2(B))$ and $\kappa_h \varepsilon L(V_h, L^2(B))$ defined by the following mappings :

$$u = (u_1, u_2) \mapsto \bar{\omega} u = (u_1, u_2, \frac{\partial u_1}{\partial x_1}, \frac{\partial u_2}{\partial x_1}, \frac{\partial u_1}{\partial x_2}, \frac{\partial u_2}{\partial x_2})$$

$$u_h = (u_{1h}, u_{2h}) \mapsto q_h u_h$$
$$= (u_{1h}|B, u_{2h}|B, \nabla_1 u_{1h}|B, \nabla_1 u_{2h}|B, \nabla_2 u_{1h}|B, \nabla_2 u_{2h}|B) ,$$
$$u_h = (u_{1h}, u_{2h}) \mapsto \kappa_h u_h = (u_{1h}\ B, u_{2h}|B)$$

where u_{ih} is considered to be defined on the whole R^2 and its restrictions to the domain B is denoted by $u_{ih}|B$.

Consider the V_h-valued piecewise constant function $u_h(t)$ on the interval $[0,T)$ defined by the relation ;

$$u_h(t) = u_h^m \quad \text{if} \quad t \varepsilon [mk, (m+1)k),\ m = 0,1,\ldots,N-1 \quad .$$

Using these concepts, we can interprete Theorem 2 as follows.

Theorem 3.

If the parameters ε, h, n and k satisfy the conditions (13),(14) and (26), then the families of functions, $\{q_h u_h\}$, $\{\sqrt{n}\,\kappa_h \chi_h u_h\}$ and $\{\kappa_h u_h\}$, remain bounded in the space $L^2(0,T;L^2(B))$, and the family of functions, $\{\kappa_h u_h\}$, remain bounded in the space $L^\infty(0,T;L^2(B))$. Namely, $q_h u_h$, $\kappa_h u_h$, and $\sqrt{n}\kappa_h \chi_h u_h$ are $L^2(0,T;L^2(B))$-stable and $\kappa_h u_h$ is $L^\infty(0,T;L^2(B))$-stable.

§4. The convergence of the approximation

The convergence of the approximate solution $u_h(t)$ to the solution of Problem 1 will be shown in this section.

Namely we have the following result.

Theorem 4.

There exists a function $w \in L^\infty(0,T;L^2(B)) \cap L^2(0,T;H_0^1(B))$ such that

(30) $\kappa_h u_h \to w$ in $w^* -L^\infty(0,T;L^2(B))$,

(31) $q_h u_h \to \phi$ in $w -L^2(0,T;L^2(B))$,

where $\phi = \overline{\omega} w$, as the set of parameters (h,k,ε,n) satisfying the conditions (13), (14) and (26), tends to $(0,0,0,\infty)$.

The restriction $u = w|\hat{\Omega}$ is the solution of Problem 1.

To prove this theorem, first we extract the subsequence from the sequence $\{u_h\}$. This subsequence, also denoted by $\{u_h\}$ for simplicity, may be assumed to satisfy the conditions (30) and (31) according to Theorem 3.

Since $\nabla_i u_{jh}$ converges to $\frac{\partial w_j}{\partial x_i}$ in the distribution sense, it holds that $\phi = \overline{\omega} w$. To complete the proof of Theorem 4, we need two lemmas.

Lemma 2. The function w satisfies the following three conditions .

(32) $w \in \hat{H}_0^1(\hat{B})$,

(33) $\operatorname{div} w = 0$,

(34) $w|\hat{E} = 0$.

Following to the treatment of J.Céa [2], we introduce the restriction operator

$\gamma_h : C_0^\infty(B) \cap V \to V_h$ defined by the relations :

(35) $\gamma_h v = v_h = (v_{1h}, v_{2h})$,

$$v_{1h}(M) = \frac{1}{h} \int_{(m_2-\frac{1}{2})h}^{(m_2+\frac{1}{2})h} v_1(m_1 h; x_2) dx_2 ,$$

$$v_{2h}(M) = \frac{1}{h}\int_{(m_1-\frac{1}{2})h}^{(m_1+\frac{1}{2})h} v_2(x_1, m_2 h)dx_1$$

This operator γ_h transforms the solenoidal function to the discretely solenoidal function. Namely, we have

(36) $$\sum_{i=1}^{2} \nabla_i v_{ih} = \nabla\cdot v_h = 0 \quad .$$

Let us define the piecewise costant functions $\psi_k(t)$, and $f_h(t)$, for $\psi\varepsilon C^\infty(0,T)$ with $\psi(T) = 0$, and for $f(t)$ in Problem 1 by the relations,

$$\psi_k(t) = \psi^m = \psi(mk) \text{ , and } f_h(t) = f_h^m \text{ , if } t\ [mk,(m+1)k),$$

respectively.

Lemma 3 Fix $v\varepsilon C_0^\infty(B)\cap V$ and $\psi\varepsilon C^\infty(0,T)$ with $\psi(T) = 0$. Assume that $\text{supp}(\psi v)\subset\hat{\Omega}$, then we have the following relations,

(37) $$\sum_{m=1}^{N} k(u_h^m, \frac{\psi^m - \psi^{m-1}}{k} v_h) = \int_0^T (u_h(t+k), \frac{\psi_k(t+k) - \psi_k(t)}{k} v_h)dt$$

$$\longrightarrow \int_0^T (w,\psi' v)dt \quad ,$$

(38) $$\nu\sum_{m=1}^{N} k((u^{m-1}, \psi^{m-1}v_h)) = \nu\int_0^T ((u_h(t),\psi_k(t)v_h))\, dt$$

$$\longrightarrow \nu\int_0^T ((w,\psi v))\, dt \quad ,$$

(39) $$(u_h^0,\psi(0)v_h) \to (u_0,\psi(0)v) \quad ,$$

(40) $$\sum_{m=1} k(f^m,\psi^{m-1}v_h) = \int_0^T (f_h,\psi_k v_h)dt$$

$$\longrightarrow \int_0^T (f,\psi v)dt \quad ,$$

(41) $$\sum_{m=1}^{N} k(\overline{\nabla}p^m,\psi^m v_h) = -\sum_{m=1}^{N} k(p^m,\psi^m\nabla\cdot v_h) = 0 \quad ,$$

(42) $$\sum_{m=1}^{N} k(n\chi_h^m u_h^m,\psi^m v_h) = 0 \quad ,$$

$$(43)\quad \sum_{m=1}^{N} kb_h(u_h^{m-1},u_h^{m-1},\psi^{m-1}v_h) = \int_0^T b_h(u_h,u_h,\psi_k v_h)dt \longrightarrow \int_0^T b(u,u,\psi v)dt \quad ,$$

(Proof of Theorem 4). According to lemma 4.5 of Fujita-Sauer, lemma 2 implies that $u = w|\hat{\Omega}\varepsilon H_\sigma^1(\hat{\Omega})$. Since u_h is the solution of (9), we have the following equality (44), noticing the relations (41) and (42) .

$$(44)\quad - \int_0^T (u_h(t+k), \frac{\psi_k(t+k) - \psi_k(t)}{k} v_h)dt + \nu\int_0^T ((u_h,\psi_k v_h)) dt + \int_0^T b_h(f_h,\psi_k v_h)dt = (u_h^0,\psi(0)v_h) + \int_0^T (f_h,\psi_k v_h)dt \quad .$$

Passing to the limit process, we have the following equality (45) by lemma 3.

$$(45)\quad - \int_0^T (u,\psi' v)dt + \nu\int_0^T ((u,\psi v)) dt + \int_0^T b(u,u,\psi v)dt = (u_0,\psi(0)v) + \int_0^T (f, v)dt \quad .$$

Since $C_0^\infty(B)\otimes \{\psi\varepsilon C_0^\infty(0,T),\psi(T) = 0\}$ is dense in $\hat{D}_\sigma(\hat{\Omega})$, the equality (45) implies that u is a solution of Problem 1. The convergence of the whole sequence is followed from the uniqueness of this solution.

(Proof of Lemma 2)

From Theorem 3, we have

$$(46)\quad \kappa_h\chi_h u_h \longrightarrow 0 \quad \text{in} \quad L^2(0,T;L^2(B)) \quad .$$

On the other hand, the definition of χ_h implies

$$\chi_h \to \chi \quad \text{a.e.} \quad (t,x) \quad .$$

Using the Lebesgue's bounded convergence theorem, we obtain

(47) $\kappa_h \chi_h u_h \to \chi w$ in $w\text{-}L^2(0,T;L^2(B))$.

From the relation (46) and (47), it follows that

$$\chi w = 0 \quad , \text{ namely } \quad w|\hat{E} = 0 \quad .$$

Let $\psi^m = \psi(mk)$ for $\psi \varepsilon C^\infty(0,T)$ with $\psi(T) = 0$, and let $v_h = \sum_{M \varepsilon B_h \cap R_h} v(M) w_{hM}$ for $v \varepsilon C_0^\infty(B)$. From the equality (10), it follows that

$$\varepsilon \sum_{m=1}^{N-1} k\left(\frac{p_h^{m+1} - p_h^m}{k}, \psi^m v_h\right) = -\varepsilon \sum_{m=1}^{N} \left(p_h^m, \frac{\psi^m - \psi^{m-1}}{k} v_h\right)$$

$$= \sum_{m=1}^{N} k(\nabla \cdot u_h, \psi^m v_h) \quad .$$

The last expression of this equality converges to

$$\int_0^T (\mathrm{div}\, w, \psi v) dt \quad \text{as } (\varepsilon,k,h,n) \text{ tends to } (0,0,0,\infty) \ .$$

On the other hand, the left hand side is majorized as follows,

$$\left|\varepsilon \sum k\left(p^m, \frac{\psi^m - \psi^{m-1}}{k} v_h\right)\right| \leqq \sqrt{\varepsilon}\left(\sum_{\ell=1}^{N} k\varepsilon |p^\ell|^2\right)^{\frac{1}{2}} \left(\sum^{N} k\left|\frac{\psi^m - \psi^{m-1}}{k} v_h\right|^2\right)^{\frac{1}{2}}$$

$$\leqq C_1 \sqrt{\varepsilon} T \left(\sum_{m=1}^{N} k\left|\frac{\psi^m - \psi^{m-1}}{k} v_h\right|^2\right)^{\frac{1}{2}} \quad .$$

Therefore

$$\int_0^T (\mathrm{div}\, w, \psi v) dt = 0 \quad .$$

This implies that $\mathrm{div}\, w = 0$ in the distribution sense.

Since our subsequence $\{u_h\}$ satisfies (31), it holds that $\mathrm{div}\, w \varepsilon L^2(0,T;L^2(B))$. So we have proved Lemma 2. q.e.d.

To prove Lemma 3, we need the following lemma.

Lemma 4. Let θ be a bounded open domain in R^2 with a smooth boundary. Fix $t_0, t_1 \varepsilon [o,T]$ with $t_0 < t_1$ such that $[t_0, t_1] \times \bar{\theta} \subset \hat{\Omega}$.

For the set $\hat{\theta} = [t_0, t_1] \times \theta$; we have

(48) $\kappa_h u_h | \hat{\theta} \longrightarrow w | \hat{\theta}$ in $L^2(t_0, t_1, L^2(\theta))$.

(Proof of lemma 3) Since u_h satisfies the relations (30) and (31), the limitting processes (37), (38), (39) and (40) are valid. The equality (41) follows from the equality (36). The equality (42) follows from the assumption that supp $(\psi v) \subset \hat{\Omega}$.

To prove (43), we may assume that supp $(\psi v) \subset \hat{\theta}$ where the set $\hat{\theta}$ is defined in Lemma 4, because of the smoothness assumption on the domain $\hat{\Omega}$ stated in §1. Since $\psi_k v_h$, and $\psi_k \nabla_i v_h$, being uniformly bounded in k,h and $(t,x) \varepsilon \hat{B}$, converge to ψv and to $\psi \frac{\partial}{\partial x_i} v$, for any $(t,x) \varepsilon B$ as h and k tend to 0, Lemma 4 implies that $\psi_k v_h u_h$ and $\psi_k \nabla_i v_h\, u_h$ converge to $\psi v u$ and to $\psi \frac{\partial v}{\partial x_j} u$ respectively as h and k tend to 0.

Combining the convergence of $\psi_k v_h u$ to $\psi v u$ and the fact that ∇u_h converges to ∇w weakly in $L^2(o,T;L^2(B))$, we have

$$\int_0^T \int_B u_{ih} \nabla_i u_{jh} (\psi_k v_h)_j dxdt$$

$$\longrightarrow \int_0^T \int_B u_i \frac{\partial}{\partial x_i} u_j \, (\psi v)_j dxdt \quad .$$

Similarly combining the convergence of $\psi_h(\nabla_i v_h)u$ to $\psi \frac{\partial v}{\partial x_i} u$ and the fact that u_h converges to u strongly in $L^2(0,T;L^2(B))$, we have

$$\int_0^T \int_B u_{ih} u_{jh} \psi_k \nabla_i v_h dxdt$$

$$\longrightarrow \int_0^T \int_B u_i u_j \psi \frac{\partial v}{\partial x_i} \; dxdt \qquad .$$

Thus the conclusion (43) has been proved. q.e.d.

(Proof of Lemma 4)

For $k = T/N$ and an integer m , define the function $\chi_k^m(t)$ as

$$\chi_k^m(t) = \begin{cases} 1/k & \text{if } t \; [(m-1)k, mk) \\ 0 & \text{otherwise} \end{cases} .$$

The function $\chi_k^0(t)$ is denoted by $\chi_k(t)$.
Assume that h and k are sufficiently small, then we can find integers m and ℓ satisfying the following conditions.

$$\begin{cases} 0 \leqq m < \ell \leqq N , \\ mk \leqq t_0 < (m+1)k , \\ (\ell - 1)k < t_1 \leqq \ell k , \\ [mk, \ell k] \times \theta \subset \hat{\Omega} . \end{cases}$$

Let $C_{m,\ell}(t)$ be the characteristic function of the interval $[mk, k)$. We denote $C_{m,\ell}u_h$, $C_{m,\ell}p_h$ and $C_{m,\ell}f_h$ by $\tilde{u}_h$, $\tilde{p}_h$ and $\tilde{f}_h$, respectively. For a while, we use the conventional notation u , p, and f for $\tilde{u}_h$,$\tilde{p}_h$ and $\tilde{f}_h$ respectively. The difference schemes (9),and (10) are rewritten as follows ;

(49) $\frac{d}{dt}(\chi_k * \dot{u}, \dot{\phi}) + \nu((u,\phi)) + b_h(u,u,\phi)$

$= (f,\phi) + (u_h^m,\phi)\chi_k^m - (u_h,\phi)\chi_k$

for any $\phi \varepsilon V_h(\theta_h)$.

(50) $\varepsilon\frac{d}{dt}(\chi_k * p,\psi) + (\nabla\cdot u,\psi)$

$=\varepsilon(p_h^m,\psi)\chi_k^m - \varepsilon(p_h,\psi)\chi_k^\ell$

for any $\psi\varepsilon V_h(\theta_h)$.

By the estimate (7) and Theorem 3, there exists a V_h-valued function $g(t)$ satisfying that

(51) $b_h(u_h,u_h,\phi) = ((g(t),\phi))$ for $\phi\varepsilon V_h$,

and that

(52) $\| g(t)\| \leqq C\| u_h(t)\|^2, \quad 0 \leqq t \leqq T$,

where C is a constant independent of ε, k, h and n. By the Fourier transformation with respect to t ,the equalities (49) and (50) become

(53) $i\tau(\hat{\chi}_k\hat{u},\phi) + \nu((\hat{u},\phi)) + ((\hat{g},\phi)) - (\hat{p},\nabla\cdot\phi)$

$= (\hat{f},\phi) + (u_h^m,\phi)\hat{\chi}_k^m - (u_h^\ell,\phi)\hat{\chi}_k^\ell$,

and

(54) $i\tau\varepsilon(\hat{\chi}_k\hat{p},\psi) + (\nabla\cdot u,\psi)$

$=\varepsilon(p_h^m,\psi)\hat{\chi}_k^m - \varepsilon(p_h^\ell,\psi)\hat{\chi}_k^\ell$,

where the symbol ^ means the Fourier image.

Taking $\phi_h = \hat{\chi}_k\hat{u}$ in the inequality (53) and $\psi_h = \hat{\chi}_k\hat{p}$ in the inequality (54), and adding these two equalities, we have

(55) $i\tau\{\varepsilon|\hat{\chi}_k\hat{p}|^2 + |\hat{\chi}_k\hat{u}|^2\} + \nu((\hat{u},\hat{\chi}_k\hat{u})) + ((\hat{g},\hat{\chi}_k\hat{u}))$

$$= (\hat{f}, \hat{\chi}_k\hat{u}) + (u_h^m, \hat{\chi}_k\hat{u})\hat{\chi}_k^m - (u_h, \hat{\chi}_k\hat{u})\hat{\chi}_k^\ell + \varepsilon(p_h^m, \hat{\chi}_k\hat{p})\hat{\chi}_k^m$$
$$- \varepsilon(p_h^\ell, \hat{\chi}_k\hat{p})\hat{\chi}_k^\ell \quad .$$

Since $|\hat{\chi}_K^m(\tau)| \leqq 1$ for any τ and m,

(56) $|\tau|\{|\hat{\chi}_k\hat{u}|^2 + \varepsilon|\hat{\chi}_k\hat{u}|\}$

$$\leqq \|\hat{g}\|\,\|\hat{u}\| + \nu\|\hat{u}\|^2 + |\hat{f}|\,|\hat{u}| + (|u_h^m| + |u_h|)|u|$$
$$+ \varepsilon(|p_h^m| + |p_h^\ell|)\,|\hat{p}| \quad .$$

It is easy to deduce that for $\beta > 1/2$,there is a constant $C(\beta)$,which depends on β ,but not on ε, k, h and n, satisfying that

(57) $$\int_{-\infty}^{\infty} \frac{|\tau|}{1 + |\tau|^\beta}\{|\hat{\chi}_k\hat{u}|^2 + \varepsilon|\hat{\chi}_k\hat{p}|^2\} \leqq C(\beta) \quad .$$

On the other hand ,we can calculate as follows,

$$\int_{-\infty}^{\infty}|\hat{\chi}_k\hat{u}|^2 d\tau \leqq \int_{-\infty}^{\infty}|\hat{u}|^2 d\tau = \int_{-\infty}^{\infty}|\hat{u}|^2 d\tau$$

(by the Parseval's equality).

So Theorem 3 implies that there is a constant C independent of ε,k,h and n such that

(58) $$\int_{-\infty}^{\infty}|\hat{\chi}_k\hat{u}|^2 d\tau \leqq C \quad .$$

Noticing that it holds for $\gamma \in (0,\frac{1}{4})$,

$$|\tau|^{2\gamma} \leqq C(\gamma)\ \frac{1 + |\tau|}{1 + |\tau|^\beta} \quad , \qquad -\infty < \tau < \infty \quad ,$$

where $C(\gamma)$ is the constant dependent on γ , we can conclude from the estimates (57) and (58) that

(59) $$\int_{-\infty}^{\infty}(1 + |\tau|^{2\gamma})|\hat{\chi}_k\hat{u}_h|^2 d\tau \leqq C \quad ,$$

where C is a constant which does not depend on ε,k,h and n.

Denote $C_{m,\ell}q_h(\chi_k * \tilde{u}_h)$ by U_h, and $C_{m,\ell}\kappa_h(\chi_h * \tilde{u}_h)$ by W_h. The estimates (57) and (59) imply that the families $\{U_h\}$ and $\{W_h\}$ are bounded sets in $L^2(R;L^2(\theta)^6)$ and $L^2(R;L^2(\theta)^2)$, respectively. So we extract the subsequences, which are still denoted by $\{U_h\}$ and $\{W_h\}$, such that U_h converges to weakly to a function U in $L^2(R;L^2(\theta)^6)$, and that W_h converges weakly to a function W in $L^2(R,L^2(\theta)^2)$. It is easy to see that $U = \bar{\omega}W$.

For any $\psi \in L^2(t_0,t_1;L^2(\theta)^2)$, let

$$\tilde{\psi} = \begin{cases} \psi & \text{if } t \in [t_0,t_1] \\ 0 & \text{otherwise} \end{cases}$$

Then it holds

$$\int_{t_0}^{t_1} (W_h, \tilde{\psi}(t))dt = \int_{mk}^{\ell k} (u_h, \bar{\chi}_k * \tilde{\psi})dt ,$$

where $\bar{\chi}_k(t) = \chi_k(-t)$.

Passing to the limit in this equality, we have

$$\int_{t_0}^{t_1} (W,\psi)dt = \int_{t_0}^{t_1} (w,\psi)dt .$$

This implies that W=w. The only remaining thing to prove is the strong convergence of W_h to W in $L^2(t_0,t_1;L^2(\theta)^2)$. To do so, it suffices to show that

$$(60) \qquad I = \int_{t_0}^{t_1} |\chi_k * \tilde{u} - w|^2 dt \longrightarrow 0 ,$$

since we have the estimates

$$\int_{t_0}^{t_1} |\chi_k * \tilde{u}_h - u_h|^2 dt \leqq \frac{k}{3}\{|u_h^m|^2 + |u_h^\ell|^2 + \sum_{i=m}^{\ell} |u_h^i - u_h^{i-1}|^2\}.$$

We can estimate the integral I as follows.

$$I = \int_{-\infty}^{\infty} |W_h - W|^2 d\tau = \int_{-\infty}^{\infty} |\hat{W}_h - \hat{W}|^2 d\tau$$

$$= \int_{|\tau| \geqq R} (1 + |\tau|^{2\gamma})^{-1} (1 + |\tau|^{2\gamma}) |\hat{W}_h - \hat{W}|^2 d\tau + \int_{-R}^{R} |\hat{W}_h - \hat{W}|^2 d\tau$$

$$\leqq C(1 + R^{2\gamma})^{-1} + \int_{-R}^{R} |W_h - W|^2 d\tau \quad .$$

This inequality holds for $\gamma \varepsilon (0, \frac{1}{4})$ by the estimate (59). The weak convergence of U_h to U implies that $\hat{U}_h(\tau)$ converges weakly to $\hat{U}(\tau)$ in $L\ (\theta)^6$ for any τ. Using the compactness argument due to Raviart (see Th.9.1 of [4]), we can conclude that

(61) $\quad \hat{W}_h(\tau) \longrightarrow \hat{W}(\tau)$ in $L^2(\theta)$ for any τ.

On the other hand for any $\psi \varepsilon L^2(\theta)$, we have

$$|(\hat{U}_h(\tau), \psi)| = | \int_{-\infty}^{\infty} (U_h, \psi \exp(-2\pi i t \tau)) dt$$

$$\leqq \left(\int_{-\infty}^{\infty} |U_h|^2 d\tau \right)^{\frac{1}{2}} |\psi| \sqrt{T}$$

$$\leqq C |\psi| \quad ,$$

where C is a constant independent of h.

The last inequality follows from the estimate (59). Therefore $|\hat{U}_h(\tau)|$ is uniformly bounded, which in turn implies that

(62) $\quad |\hat{W}_h(\tau) - \hat{W}(\tau)|$ is bounded uniformly in τ.

Therefore, by (61),(62) and the Lebesgue's bounded convergence Theorem, it holds that $\int_{-R}^{R} |\hat{W}_h - \hat{W}|^2 d\tau$ tends to 0 as (k, h, ε, n) tends to $(0, 0, 0, \infty)$. Thus we have conclusion (60). q.e.d.

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A Finite Element Approximation Corresponding to the Upwind Finite Differencing

By

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§1. Introduction

Lately it has been requested to solve numerically the diffusion equations with drift terms (the first derivative terms with respect to spatial variables) in a large domain in relation to the problems of water pollution in coastal seas, of surface discharge of heated water of atomic plants, of convection currents in a horizontal layer of fluid, and so on. In these fields the finite element method is preferred to the finite difference method. This is partly because the former has the pretty wide flexibility with respect to the choice of the position of nodal points, which is effective especially in the case where the considered domain is not a simple figure.

When the ratio of the velocity of the drift to the diffusion constant is small, they are solved easily by the standard finite element method. However, in the case where its ratio is large, the L^∞-stability condition forces us to take very small elements. Although the same difficulty arises when the central finite difference is used to approximate the drift terms, it can be overcome by the use of the upwind difference approximation.

In this paper we propose a finite element approximation corresponding to the upwind differencing. Using this approximation, we obtain the L^∞-stability condition which does not require that

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elements should be small, and then prove the convergence of the numerical solutions to the exact one.

It often arises that an approximate solution which has negative parts is of no use from the physical point of view, for example, when the solution indicates temperature or density. Meanwhile it is shown that the L^∞-stability implies the nonnegativity of numerical solutions in an appropriate situation (see Corollaries 1 and 2). This is the reason why we esteem the L^∞-stability.

For the stationary equation of the one we consider, Kikuchi [3] showed the discrete maximum principle by introducing the artificial viscosity term. His method is applicable to the nonstationary problem, but it requires that all the angle of triangular elements are strictly less than $\pi/2$. In our method, $\pi/2$ is allowable and it is considered that this makes triangulation of the domain pretty easy.

§2. Preliminaries

Let Ω be a polygonal domain in $\mathbb{R}^2$, Γ be its boundary, and T be a fixed positive number. We consider the following problem,

$$(2.1)\quad \left\| \begin{array}{ll} \frac{\partial u}{\partial t} = d\Delta u - (v\cdot\nabla)u + f & \text{in } Q = \Omega\times(0,T), \\ u = 0 & \text{on } \Sigma = \Gamma\times(0,T), \\ u = u^0 & \text{in } \Omega \quad \text{at } t=0, \end{array} \right.$$

where d is a positive constant, $v = (v_1(x,y),\ v_2(x,y))$ or $(v_1(x,y,t),\ v_2(x,y,t))$, $u^0 = u^0(x,y)$ and $f = f(x,y,t)$ are given continuous functions, and

$$v\cdot\nabla = v_1\frac{\partial}{\partial x} + v_2\frac{\partial}{\partial y} .$$

In our problem v is not so small in comparison with d.

We triangulate $\bar{\Omega}$ to obtain a set of closed triangles $\{ T_j \}_{j=1}^{N_E}$ and a set of interior nodal points $\{ P_i \}_{i=1}^{N}$, holding the usual assumption that triangles do not degenerate. By interior nodal points we mean vertices existing in Ω. Define κ, h and V_h as follows:

κ = the mimimum perpendicular length of all the triangles,

h = the maximum side length of all the triangles,

and

$$V_h = \{ \phi_h \; ; \; \phi_h \in C(\bar{\Omega}), \text{ linear on each triangle, and } \phi_h = 0 \text{ on } \Gamma \}.$$

With each interior nodal point P_i, we associate functions ϕ_{ih} and $\bar{\phi}_{ih}$ satisfying the following properties,

i) $\phi_{ih} \in V_h$ and $\phi_{ih}(P_j) = \delta_{ij}$ for i,j = 1,...,N,

and

ii) $\bar{\phi}_{ih} \in L^2(\Omega)$, and $= 1$ on S_i, and $= 0$ otherwise, where S_i is the barycentric domain associated with P_i (see Fig. 1 and [2]).

Define a lumping operator - from V_h into $L^2(\Omega)$ as follows :

$$- : V_h \to L^2(\Omega),$$
$$u_h \mapsto \bar{u}_h = \Sigma_{i=1}^{N} u_i \bar{\phi}_{ih},$$

where u_i is the value of u_h at P_i.

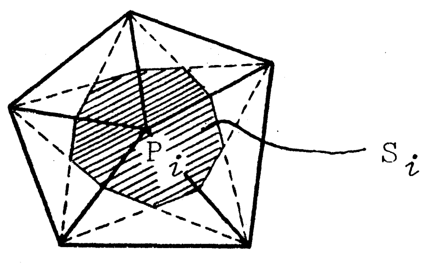

Fig. 1. Barycentric domain S_i associated with P_i.

Now the standard explicite finite element approximation scheme (of lumped mass type) is as follows:

$$
(2.2)\quad \left\{\begin{array}{l}
\text{Find } \{ u_h^n \}_{n=1,..N_T} \subset V_h \text{ such that} \\
\left(\dfrac{\bar{u}_h^{n+1} - \bar{u}_h^n}{\tau}, \bar{\phi}_h\right) = -d\, a(u_h^n, \phi_h) - ((v\cdot\nabla)u_h^n, \phi_h) + (f(n\tau), \phi_h) \\
\qquad\qquad\qquad\qquad \text{for all } \phi_h \in V_h, \quad n = 0,\dots,N_T-1 , \\
u_h^0(P_j) = u^0(P_j) \qquad \text{for } j = 1,\dots,N,
\end{array}\right.
$$

where τ is a time mesh, $N_T = [\frac{T}{\tau}]$, and

$$
a(u,v) = \int_\Omega \{ \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y} \}\, dxdy .
$$

To establish the L^∞-stability of (2.2) we must employ the triangulation of strictly acute type, i.e., all the angles of triangles are less than or equal to $\pi/2 - \varepsilon$, where ε is small positive constant. Then, the L^∞-stability conditions for (2.2) are

$$
(2.3)\qquad \tau \leqq \frac{1}{3d}\kappa^2 ,
$$

and

$$
(2.4)\qquad h \leqq \frac{3\tan\varepsilon}{\sin(\frac{\pi}{2} - \varepsilon)} \frac{d}{|v|} ,
$$

where $|v| = \{v_1^2 + v_2^2\}^{1/2}$.

In actual problems of water pollution in coastal seas,

$$
d = 1 \sim 10 \text{ m}^2/\text{sec} \quad \text{and} \quad |v| = 0.5 \sim 2 \text{ m/sec}
$$

and, even in the pretty fine subdivision, h = 100 ~ 1000 m. From this example we can see that condition (2.4) is very severe in the practical computation. In our method, although condition (2.3) becomes a little restrictive, we can get rid of condition (2.4) and allow the triangulation of (not strictly) acute type.

We use the following notations throughout this paper:

$$
\langle i,j\rangle = \{i, i+1, i+2, .., j\} \quad \text{for integers } i < j ,
$$

$$
(u,v) = \int_\Omega u(x,y)v(x,y)\, dxdy \quad \text{for } u,v \in L^2(\Omega),
$$

$$\| u \|_0 = \{(u, u)\}^{1/2} ,$$

$$\| u \|_A = \{\| \frac{\partial u}{\partial x} \|_0^2 + \| \frac{\partial u}{\partial y} \|_0^2\}^{1/2} ,$$

and

$$\| u \|_1 = \{\| u \|_0^2 + \| u \|_A^2\}^{1/2} .$$

Furthermore we use c as a generic constant, which does not depend on h, κ and τ and does not necessarily have the same value at each occurence.

§3. An Upwind Finite Element Approximation

In the present section we consider the case where $v = v(x,y)$. Here, we introduce upwind finite elements. A triangle T_j is called a x-upwind finite element at nodal point P_i if the following two conditions are satisfied:

i) P_i is a vertex of T_j,

and

ii) $T_j - \{P_i\}$ meets the oriented half line with end point P_i, which has the same direction as the x-axis if $v_1(P_i) \geqq 0$ and has the opposite direction to it if $v_1(P_i) < 0$.

A y-upwind finite element at P_i is defined by replacing x and $v_1(P_i)$ with y and $v_2(P_i)$ respectively in the above definition.

Now our upwind finite element approximation scheme of explicit type for (2.1) is as follows:

$$(3.1) \quad \left\| \begin{array}{l} \text{Find } \{ u_h^n \}_{n \in <0, N_T>} \subset V_h \quad \text{such that} \\ (\frac{\bar{u}_h^{n+1} - \bar{u}_h^n}{\tau}, \bar{\phi}_h) = -d\, a(u_h^n, \phi_h) + (R(u_h^n), \bar{\phi}_h) + (\bar{f}(n\tau), \bar{\phi}_h) \\ \qquad\qquad \text{for all } \phi_h \in V_h, \quad n \in <0, N_T - 1>, \\ u_h^0(P_j) = u^0(P_j) \qquad \text{for } j \in <1, N>, \end{array} \right.$$

where

$$\bar{f}(n\tau) = \Sigma_{i=1}^{N} f_i^n \bar{\phi}_{ih} , \qquad f_i^n = f(P_i, n\tau),$$

$$R(u_h^n) = \Sigma_{i=1}^{N} R_i(u_h^n) \bar{\phi}_{ih} ,$$

$$R_i(u_h^n) = -v_1(P_i)\frac{\partial u_h^n}{\partial x} \Big|_{T_x^i} - v_2(P_i) \frac{\partial u_h^n}{\partial y} \Big|_{T_y^i} ,$$

T_x^i is a x-upwind finite element at P_i,

and

T_y^i is a y-upwind finite element at P_i.

Note that, if there exists two x-upwind (or y-upwind) finite elements at P_i, we choose an arbitrary fixed one of them as T_x^i (or T_y^i).

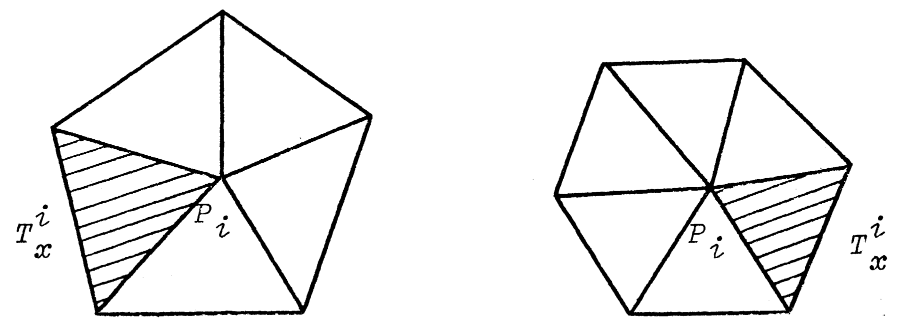

Fig. 2. X-upwind finite elements T_x^i when $v_1(P_i) \geqq 0$ (left), and < 0 (right).

Now we show the L^∞-stability condition of (3.1).

Theorem 1. *Assume the triangulation is of acute type and that*

(3.2) $$\tau \leqq \frac{\kappa^2}{3d + V\kappa} ,$$

where

(3.3) $$V = \max_{(x,y)\in\bar{\Omega}} (|v_1(x,y)| + |v_2(x,y)|).$$

Then, scheme (3.1) is L^∞-stable and it holds that

(3.4) $$\min_{(x,y)\in\bar\Omega} u^0 + T \min_{(x,y)\in\bar Q} f \leqq u_h^n(x,y) \leqq \max_{(x,y)\in\bar\Omega} u^0 + T \max_{(x,y)\in\bar Q} f$$

for $n\in\langle 0,N_T\rangle$, $(x,y)\in\bar\Omega$.

Proof. We begin by proving the following inequality

(3.5) $$\min_{j\in\langle 1,N\rangle} u_j^n + \tau \min_{j\in\langle 1,N\rangle} f_j^n \leqq u_i^{n+1} \leqq \max_{j\in\langle 1,N\rangle} u_j^n + \tau \max_{j\in\langle 1,N\rangle} f_j^n$$

for $i \in \langle 1,N\rangle$, $n \in \langle 0,N_T-1\rangle$.

Fix an interior nodal point P_i arbitrarily. Substituting $\phi_h=\phi_{ih}$ in (3.1), we have

(3.6) $$u_i^{n+1} = \{u_i^n - \frac{\tau d}{M_{ii}} \Sigma_{j=1}^N a(\phi_{jh}, \phi_{ih}) u_j^n\} + \tau R_i(u_h^n) + \tau f_i^n ,$$

where $M_{ii}= (\bar\phi_{ih}, \bar\phi_{ih})$. Here, we consider only the case where $v_1(P_i), v_2(P_i) \geqq 0$ and P_i has neighboring nodal points $\{P_{i_1},\ldots,P_{i_6}\}$ since, in the other cases, we can prove (3.6) in the same way.

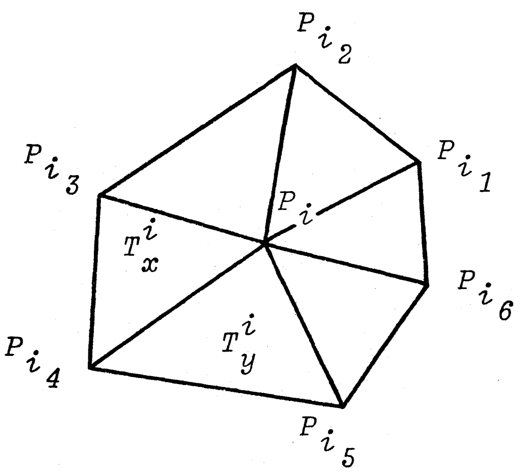

Fig. 3. P_i and its neighboring nodal points.

In this case T_x^i is $\Delta P_i P_{i_3} P_{i_4}$ and T_y^i is $\Delta P_i P_{i_4} P_{i_5}$. By a brief calculation we obtain

$$(3.7)\qquad R_i(u_h^n) = -v_1(P_i)\left\{\frac{y_{i_3} - y_{i_4}}{2M_{ix}} u_i^n + \frac{y_{i_4} - y_i}{2M_{ix}} u_{i_3}^n + \frac{y_i - y_{i_3}}{2M_{ix}} u_{i_4}^n\right\}$$

$$-v_2(P_i)\left\{\frac{x_{i_5} - x_{i_4}}{2M_{iy}} u_i^n + \frac{x_{i_4} - x_i}{2M_{iy}} u_{i_5}^n + \frac{x_i - x_{i_5}}{2M_{iy}} u_{i_4}^n\right\},$$

where M_{ix} = the area of T_x^i,

M_{iy} = the area of T_y^i,

and (x_j, y_j) the coordinates of P_j. Substituting (3.7) in (3.6), we get

$$(3.8)\qquad u_i^{n+1} = \{1 - \tau(\frac{d}{M_{ii}} a_{ii} + b_{ii})\}u_i^n + \Sigma_{k=1}^{6} \tau(-\frac{d}{M_{ii}} a_{i_k i} + b_{i_k i})$$

$$\times u_{i_k}^n + \tau f_i^n,$$

where

$$a_{ji} = a(\phi_{jh}, \phi_{ih}),$$

$$b_{ii} = v_1(P_i)\frac{y_{i_3} - y_{i_4}}{2M_{ix}} + v_2(P_i)\frac{x_{i_5} - x_{i_4}}{2M_{iy}},$$

$$b_{i_3 i} = -v_1(P_i)\frac{y_{i_4} - y_i}{2M_{ix}},$$

$$b_{i_4 i} = -v_1(P_i)\frac{y_i - y_{i_3}}{2M_{ix}} - v_2(P_i)\frac{x_i - x_{i_5}}{2M_{iy}},$$

$$b_{i_5 i} = -v_2(P_i)\frac{x_{i_4} - x_i}{2M_{iy}},$$

and

$$b_{i_1 i} = b_{i_4 i} = b_{i_6 i} = 0.$$

From the way of the choice of upwind finite elements we have

$$b_{i_k i} \geqq 0 \qquad \text{for all k.}$$

leanwhile, it holds that

$$a_{ji} \leqq 0 \qquad \text{for } i \neq j,$$

ecause the triangulation is of acute type (see [1]). Hence, the oefficients of $u_{i_k}^n$ in (3.8) are nonnegative. As for the coefficient f u_i^n, we have

$$1 - \tau\left(\frac{d}{M_{ii}} a_{ii} + b_{ii}\right) \geqq 1 - \tau\left(\frac{3d}{\kappa^2} + \frac{V}{\kappa}\right) \geqq 0 ,$$

sing the estimate in [2]

$$\frac{a_{ii}}{M_{ii}} \leqq \frac{3}{\kappa^2} .$$

oticing that the sum of all the coefficients of u_i and u_{i_k} is equal o identity, we obtain (3.5).

From (3.5) we have

$$(3.9) \qquad \min_{(x,y)\in\bar{\Omega}} u_h^n + \tau \min_{(x,y)\in\bar{\Omega}} f(n\tau) \leqq \min_{(x,y)\in\bar{\Omega}} u_h^{n+1} \leqq \max_{(x,y)\in\bar{\Omega}} u_h^{n+1}$$

$$\leqq \max_{(x,y)\in\bar{\Omega}} u_h^n + \tau \max_{(x,y)\in\bar{\Omega}} f(n\tau)$$

$$\text{for} \qquad n \in \langle 0, N_T-1 \rangle,$$

hich implies (3.4).

Corollary 1. *Assume the same assumption as Theorem 1. If* $\geqq 0$*, and* $u^0 \geqq 0$*, then*

$$u_h^n(x,y) \geqq 0 \qquad \text{for } (x,y) \in \bar{\Omega}, \quad n \in \langle 0, N_T \rangle.$$

Proof. This result is lead from (3.8) because all the coefficients of u_j are nonnegative.

Now we proceed with the derivation of the error estimates.

Theorem 2. *Suppose that the exact solution $u \in C^2(\bar{Q})$ and that $f \in C^1(\bar{Q})$. Then, under the same conditions as Theorem 1, we have the following estimates,*

$$\max_{n\in<0,N_T>} \|\bar{u}_h^n - u(n\tau)\|_0 \ , \ \{\tau\Sigma_{n=0}^{N_T} \|u_h^n - u(n\tau)\|_A^2\}^{1/2} \leq ch.$$

For the proof of Theorem 2 we need the following lemmas.

Lemma 1. *Suppose the same conditions as Theorem 1 and that $u^0 \in C^2(\bar{\Omega})$. Then u_h^n, the solution of (3.1), satisfies that*

$$(3.10) \qquad \|u_h^n\|_A \leqq \left\|\frac{u_h^{n+1} + u_h^n}{2}\right\|_A + c\kappa \qquad \text{for } n \in <0,N_T-1>.$$

Proof. We first prove the following inequality

$$(3.11) \qquad \max_{j\in<1,N>,\ n\in<0,N_T-1>} \left|\frac{u_j^{n+1} - u_j^n}{\tau}\right| \leqq c\ .$$

Put $s_j^n = \dfrac{u_j^{n+1} - u_j^n}{\tau}$, and s_j^n satisfies

$$s_i^{n+1} = \{s_i^n - \frac{\tau d}{M_{ii}} \Sigma_{j=1}^N a(\phi_{jh},\phi_{ih})\ s_j^n\} + \tau\ R_i(\frac{u_h^{n+1} - u_h^n}{\tau})$$
$$+ \tau\frac{\partial f}{\partial t}(P_i,\ n\tau+\theta\tau)$$

for $i \in <1,n>,\ n \in <0,N_T-1>$ and $\exists\theta \in (0,1)$.

Applying Theorem 1, we have

$$(3.12) \qquad \max_{j\in<1,N>} |s_j^n| \leqq \max_{j\in<1,N>} |s_j^0| + T \max_{(x,y,t)\in\bar{Q}} \left|\frac{\partial f}{\partial t}\right|$$

for $n \in <0,N_T>$.

By the definition it holds

$$(3.13) \qquad s_i^0 = -\frac{d}{M_{ii}} \sum_j a(\phi_{jh}, \phi_{ih}) u_j^0 + R_i(u_h^0) + f_i^0 .$$

The second term of the right of (3.13) is bounded since it approximates $-(v\cdot\nabla)u^0(P_i)$. Although the first term of the right of (3.13) does not hold the local consistency, i.e., it does not approximate $\Delta u^0(P_i)$ even if h is very small, we can show the boundedness of it. Actually, expanding u_j^0 at P_j, we have

$$\begin{aligned}
\sum_j a(\phi_{jh}, \phi_{ih})u_j^0 &= \sum_j a(\phi_{jh}, \phi_{ih})\{ u_i^0 + (x_j - x_i)\frac{\partial u_i^0}{\partial x} \\
&\qquad + (y_j - y_i)\frac{\partial u_i^0}{\partial y} + O(h_i^2) \} \\
&= (u_i^0 - x_i\frac{\partial u_i^0}{\partial x} - y_i\frac{\partial u_i^0}{\partial y})a(1, \phi_{ih}) + \frac{\partial u_i^0}{\partial x} a(x, \phi_{ih}) \\
&\quad + \frac{\partial u_i^0}{\partial y} a(y, \phi_{ih}) + O(h_i^2) \\
&= O(h_i^2),
\end{aligned}$$

where

h_i = the maximum side length of the triangles whose vertices include P_i.

Since it is obvious that

$$M_{ii} \geqq ch_i^2,$$

we obtain the boundedness of the first term of (3.13). Thus, (3.11) is valid.

Now from (3.11) it follows that

$$\begin{aligned}
\|u_h^n\|_A &\leqq \left\|\frac{u_h^{n+1} + u_h^n}{2}\right\|_A + \left\|\frac{u_h^{n+1} - u_h^n}{2}\right\|_A \\
&\leqq \left\|\frac{u_h^{n+1} + u_h^n}{2}\right\|_A + \frac{c}{\kappa} \|u_h^{n+1} - u_h^n\|_0
\end{aligned}$$

$$\leqq \left\| \frac{u_h^{n+1} + u_h^n}{2} \right\|_A + \frac{c\tau}{\kappa}$$

$$\leqq \left\| \frac{u_h^{n+1} + u_h^n}{2} \right\|_A + c\kappa .$$

This completes the proof of Lemma 1.

Lemma 2. *Suppose* $\{ w_h^n \}_{n \varepsilon <0,N_T>} \subset V_h$ *satisfies the following two relations:*

i) *For all* $\psi_h \ \varepsilon \ V_h$ *and* $n \ \varepsilon \ <0,N_T-1>$ *it holds that*

$$(3.14) \qquad \left(\frac{\bar{w}_h^{n+1} - \bar{w}_h^n}{\tau} , \bar{\psi}_h \right) = -d \ a(w_h^n, \psi_h) + (R(w_h^n), \bar{\psi}_h) + ch\theta(n\tau)\|\psi_h\|_1$$

where $\theta = \theta(t)$ *is a bounded function such that* $|\theta| \leqq 1$.

ii) *For* $n \ \varepsilon \ <0,N_T-1>$ *it holds that*

$$(3.15) \qquad \|w_h^n\|_A \leqq \left\| \frac{w_h^{n+1} + w_h^n}{2} \right\|_A + c\kappa .$$

Then, under the condition $\tau < \frac{\kappa^2}{3d}$ *, we have the following estimates,*

$$(3.16) \qquad \max_{n \ \varepsilon \ <0,N_T>} \|\bar{w}_h^n\|_0 \ , \ \{\tau \sum_{n=0}^{N_T} \|w_h^n\|_A^2\}^{1/2} \leqq c\{\|\bar{w}_h^0\|_0 + h\}.$$

Proof. We substitute in (3.14) $\psi_h = w_h^{n+1} + w_h^n$ and then after a brief calculation, we obtain

$$(3.17) \qquad \|\bar{w}_h^{n+1}\|_0^2 - \|\bar{w}_h^n\|_0^2 = - \frac{\tau d}{2} \|w_h^{n+1} + w_h^n\|_A^2 + \frac{\tau d}{2} (\|w_h^{n+1}\|_A^2 - \|w_h^n\|_A^2)$$

$$+ \tau(R(u_h^n), \bar{w}_h^{n+1} + \bar{w}_h^n) + \tau ch\theta\|w_h^{n+1} + w_h^n\|_1 .$$

Since v is continuous in $\bar{\Omega}$, it is shown easily that

$$|(R(w_h^n), \bar{w}_h^{n+1} + \bar{w}_h^n)| \leqq c\|w_h^n\|_A(\|\bar{w}_h^{n+1}\|_0 + \|\bar{w}_h^n\|_0).$$

Applying the Young's inequality to (3.17), we have

$$\|\bar{w}_h^{n+1}\|_0^2 - \frac{\tau d}{2} \|\bar{w}_h^{n+1}\|_A^2 + \tau(d/2 - \varepsilon - \varepsilon')\|w_h^n\|_A^2$$

$$(3.18) \qquad \leqq \|\bar{w}_h^n\|_0^2 - \frac{\tau d}{2} \|\bar{w}_h^n\|_A^2 + c(\varepsilon)\tau h^2 + c(\varepsilon')\tau\{\|\bar{w}_h^{n+1}\|_0^2 + \|\bar{w}_h^n\|_0^2\},$$

where ε and ε' are positive constants which are fixed so small that $d/2 - \varepsilon - \varepsilon' > 0$, and $C(\varepsilon)$ and $C(\varepsilon')$ are constants depending on ε and ε' respectively. Using the following inequality in [2],

$$\|w_h\|_A \leqq \frac{\sqrt{6}}{\kappa} \|\bar{w}_h\|_0 ,$$

and summing (3.18) from $n = 0$ to $n-1$, we obtain (3.16) by the Gronwall's inequality.

Proof of Theorem 2. We begin by proving that u satisfy the equation

$$(3.19) \qquad \left(\frac{\bar{u}^{n+1} - \bar{u}^n}{\tau}, \bar{\psi}_h \right) = -d\, a(u^n, \psi_h) + (R(u^n), \bar{\psi}_h) + (f(n\tau), \psi_h)$$

$$+ ch\theta(n\tau)\|\psi_h\|_1 \quad \text{for all} \quad \psi_h \in V_h ,$$

where

$$u^n = \Sigma_{j=1}^N u(P_j, n\tau)\, \phi_{jh}, \qquad \bar{u}^n = \Sigma_{j=1}^N u(P_j, n\tau)\, \bar{\phi}_{jh} ,$$

and

$\theta = \theta(t)$ is a bounded function such that $|\theta| \leqq 1$.

Since u is the exact solution, it holds

$$(3.20) \qquad \left(\frac{\partial u}{\partial t}, \psi_h \right) = -d\, a(u, \psi_h) - ((v \cdot \nabla)u, \psi_h) + (f, \psi_h)$$

$$\text{for all} \quad \psi_h \in V_h.$$

We observe that, for all $\psi_h \in V_h$,

$$(3.21) \qquad \left| \left(\frac{\bar{u}^{n+1} - \bar{u}^n}{\tau}, \bar{\psi}_h \right) - \left(\frac{\partial u}{\partial t}(n\tau), \psi_h \right) \right| \leqq c(\tau + h)\|\psi_h\|_1 ,$$

(3.22) $$|a(u^n, \psi_h) - a(u(n\tau), \psi_h)| \leqq ch\|\psi_h\|_1 ,$$
and

(3.23) $$|(R(u^n), \bar{\Psi}_h) + ((v\cdot\nabla)u(n\tau), \psi_h)| \leqq ch\|\psi_h\|_1 .$$

We prove only (3.23) because the others are shown in the same way. Now,

(3.24) $$(R(u^n), \bar{\Psi}_h) + ((v\cdot\nabla)u(n\tau), \psi_h)$$
$$= (R(u^n) + (v\cdot\nabla)u(n\tau), \bar{\Psi}_h) + ((v\cdot\nabla)u(n\tau), \psi_h - \bar{\Psi}_h).$$

The second term of the right of (3.24) is bounded by $ch\|\psi_h\|_1$. Expanding $R(u^n)$ and $(v\cdot\nabla)u(n\tau)$ in a neighborhood of P_i, we have

(3.25) $$R(u^n) = \Sigma_{i=1}^{N} (-v\cdot\nabla)u(P_i, n\tau)\,\bar{\Phi}_{ih} + ch\theta_1 ,$$

and

(3.26) $$(v\cdot\nabla)u(n\tau) = \Sigma_{i=1}^{N} (v\cdot\nabla)u(P_i, n\tau)\,\bar{\Phi}_{ih} + ch\theta_2 ,$$

where θ_i $(i=1,2)$ are functions such that $|\theta_i| \leqq 1$. Using (3.25) and (3.26), we can estimate the first term of the right of (3.24) by $ch\|\bar{\Psi}_h\|_0$. Hence, we obtain (3.23). Combining (3.20)∼(3.23), we get (3.19).

Since u_h^n is a solution of (3.1), $w_h^n = u_h^n - u^n$ satisfies

(3.27) $$\left(\frac{w_h^{n+1} - w_h^n}{\tau}, \bar{\Psi}_h\right) = -d\,a(w_h^n, \psi_h) + (R(w_h^n), \bar{\Psi}_h)$$
$$+ \{(\bar{f}^n, \bar{\Psi}_h) - (f(n\tau), \psi_h)\} + ch\theta(n\tau)\|\psi_h\|_1$$
for all $\psi_h \in V_h$.

The third term of the right of (3.27) is estimated as follows,

$$|(\bar{f}^n, \bar{\Psi}_h) - (f(n\tau), \psi_h)| = |(\bar{f}^n - f(n\tau), \bar{\Psi}_h) + (f(n\tau), \bar{\Psi}_h - \psi_h)|$$
$$\leqq ch(\|f(n\tau)\|_1\|\bar{\Psi}_h\|_0 + \|f(n\tau)\|_0\|\psi_h\|_1).$$

Therefore w_h^n satisfies the condition (3.14). Applying Lemme 1 and

Lemma 2, we obtain

$$\max_{n \in \langle 0, N_T \rangle} \| \bar{u}_h^n - \bar{u}^n \|_0 , \quad \{ \tau \Sigma_{n=0}^{N_T} \| u_h^n - u^n \|_A \}^{1/2} \leqq ch .$$

This concludes the proof of Theorem 2 since it holds

$$\| \bar{u}^n - u(n\tau) \|_0 \leqq ch^2 ,$$

and

$$\| u^n - u(n\tau) \|_A \leqq ch \qquad \text{for} \quad n \in \langle 0, N_T \rangle .$$

§4. An Implicit Scheme

In this section we consider an upwind finite element approximation scheme of implicit type in the case where $v = v(x,y,t)$. Our scheme is as follows,

(4.1)

$$\left\| \begin{array}{l} \text{Find} \quad \{u_h^n\}_{n \in \langle 0, N_T \rangle} \subset V_h \quad \text{such that} \\[2ex] \left(\dfrac{\bar{u}_h^{n+1} - \bar{u}_h^n}{\tau} , \bar{\phi}_h \right) = -d\, a(u_h^{n+1}, \phi_h) + (R^{n+1}(u_h^{n+1}), \bar{\phi}_h) \\[2ex] \qquad\qquad + (\bar{f}(n\tau + \tau), \bar{\phi}_h) \\[2ex] \qquad\qquad\qquad \text{for all } \phi_h \in V_h , \quad n \in \langle 0, N_T - 1 \rangle , \\[2ex] u_h^0(P_j) = u^0(P_j) \qquad \text{for } j \in \langle 1, N \rangle , \end{array} \right.$$

where the superscript $n+1$ of R^{n+1} indicates that upwind finite elements at P_i are taken according to the signature of $v(P_i, (n+1)\tau)$.

The standard implicit finite element scheme corresponding to (4.1) is unconditionally L^2-stable but it requires condition (2.4) for the L^∞-stability. On the other hand, we can show that (4.1) is unconditionally L^∞-stable. Corresponding results to Theorem 1, Corollary 1, and Theorem 2 are as follows:

Theorem 3. *Assume the triangulation is of accute type. Then, scheme (4.1) is unconditionally* L^∞*-stable, i.e., for any* τ *and* κ *(>0), (3.4) is holds.*

Corollary 2. *Under the same condition as Theorem 3,* $f, u^0 \geqq 0$ *implies that*

$$u_h^n(x, y) \geqq 0 \quad \text{for } (x,y)\in\Omega , \; n \in \langle 0, N_T\rangle .$$

Theorem 4. *Suppose that the exact solution* $u \in C^2(\bar{Q})$ *and that* $f \in C^1(\bar{Q})$. *Under the same assumption as Theorem 3, the following estimates hold,*

$$\max_{n\in\langle 0,N_T\rangle} \|\bar{u}_h^n - u(n\tau)\|_0 , \; \{\tau\Sigma_{n=0}^{N_T}\|u_h^n - u(n\tau)\|_A^2\}^{1/2} \leqq c(h+\tau).$$

We omit the proofs of the above results because they are a slight modification of the proofs in the previous section (Theorem 4 is proved without estimate (3.10)).

§5. Concluding Remarks

Upwind finite element approximation schemes have been discussed. Our method is applicable to the first order hyperbolic equations and we can obtain easily the L^∞-stability and the L^∞-convergence if the exact solution has an appropriate smootheness. Because, in these problems, our scheme has local consistency.

In §3, we introduced two upwind finite elements at P_i, i.e., x-upwind finite element T_x^i and y-upwind finite element T_y^i. But we may use only one upwind finite element T_c^i at P_i, which is defined as triangle T_j satisfying the conditions,

i) P_i is a vertex of T_j,

and

ii) $T_j - \{P_i\}$ meets the oriented half line with end point P_i which direction is $(v_1(P_i), v_2(P_i))$.

Then, we obtain the same results (Theorems 1~4 and Corollaries 1,2) with $V = |v|$ instead of (3.3).

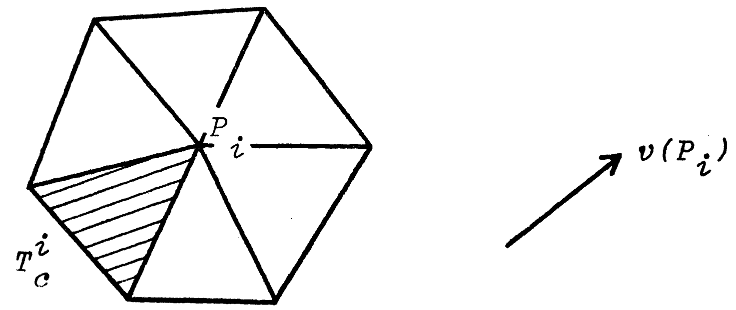

Fig. 4. Upwind finite element T_c^i at P_i.

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