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CONTENTS

On a three-point difference scheme for a singular perturbation problem without a first derivative term. I

By

Koichi Niijima

1. Introduction

Let ε be a parameter satisfying $0<\varepsilon\leq 1$ and consider the boundary value problem

$$\varepsilon y'' - b(x,\varepsilon)y = f(x,\varepsilon), \qquad 0\leq x\leq 1, \tag{1.1a}$$

$$y(0) = a_0, \quad y(1) = a_1, \tag{1.1b}$$

where $b(x,\varepsilon)$ and $f(x,\varepsilon)$ are twice continuously differentiable with respect to x on $D=\{(x,\varepsilon)\,|\,0\leq x\leq 1,\ 0<\varepsilon\leq 1\}$, and bounded there together with their first and second derivatives with respect to x. Assume further that $b(x,\varepsilon)\geq\delta>0$ in $\bar{D}$.

Recently, J.J.H.Miller [2] derived an exponentially fitted difference scheme for the problem (1.1), and showed that the solution of this scheme converges to that of (1.1) uniformly in ε with order h which denotes a mesh step.

In the present paper, we give a three-point difference scheme whose solution converges to that of (1.1) uniformly in ε with order h^2. J.J.H.Miller [2] employed the method of A.M.Il'in [1] in constructing his difference scheme and in proving the uniform convergence, but we use the Liouville-Green transformation which makes easy the error analysis.

2. Approximation to the problem (1.1)

We begin with approximating $b(x,\varepsilon)$ and $f(x,\varepsilon)$. Let us introduce the uniform mesh $x_i=ih$, $i=0,1,\ldots,N$, where $Nh=1$, and approximate the functions $b(x,\varepsilon)$ and $f(x,\varepsilon)$ in each subinterval $[x_i,x_{i+1}]$, respectively, by

$$B(x,\varepsilon) = 1/\left(\beta_i\frac{x - x_i}{h} + \alpha_i\right)^4$$

and

$$F(x,\varepsilon) = \left\{\alpha_{i+1}(\alpha_{i+1}{}^3 f_{i+1}-\alpha_i{}^3 f_i)\ \frac{x-x_i}{\beta_i(x-x_i)+\alpha_i h} +\alpha_i{}^3 f_i\right\}$$

$$\div \left(\beta_i\frac{x - x_i}{h} + \alpha_i\right)^3 ,$$

where α_i, β_i and f_i denote

$$\alpha_i = 1/\sqrt[4]{b(x_i,\varepsilon)} ,$$

$$\beta_i = \alpha_{i+1} - \alpha_i$$

and

$$f_i = f(x_i,\varepsilon),$$

respectively. We remark that such approximating functions are determined from the Liouville-Green transformation appearing later on.

Consider the following approximation problem to (1.1);

$$\varepsilon Y'' - B(x,\varepsilon)Y = F(x,\varepsilon), \qquad (2.1a)$$

$$Y(0) = a_0 , \quad Y(1)= a_1 . \qquad (2.1b)$$

Then we can prove

Lemma. Let $y(x)$ and $Y(x)$ denote solutions of (1.1) and (2.1), respectively, and let $\|\cdot\|_\infty$ denote a maximum norm. Then there is a constant C, independent of h and ε, such that

$$\|Y - y\|_\infty \leqq Ch^2 . \tag{2.2}$$

Proof. We first note that $B(x,\varepsilon)$ and $F(x,\varepsilon)$ are continuous functions on D. Define $r(x)$ by $r(x)=Y(x)-y(x)$. This $r(x)$ satisfies

$$\varepsilon r'' - b(x,\varepsilon)r = (B(x,\varepsilon) - b(x,\varepsilon))Y + F(x,\varepsilon) - f(x,\varepsilon)$$

and

$$r(0) = r(1) = 0.$$

Since $b(x,\varepsilon)\geqq\delta>0$, the maximum principle assures that for a constant C_1 independent of h and ε,

$$\|r\|_\infty \leqq C_1(\|B - b\|_\infty \|Y\|_\infty + \|F - f\|_\infty).$$

Accordingly, it suffices to show that there are constants C_2 and C_3, independent of h and ε, such that

$$\|B - b\|_\infty \leqq C_2h^2$$

and

$$\|F - f\|_\infty \leqq C_3h^2 .$$

The first estimate follows from the Taylor's theorem. Indeed,

$$B(x,\varepsilon) - b(x,\varepsilon) = -(\frac{4}{h} b_i^{5/4}\beta_i+b_i')(x-x_i) + O(h^2)$$

$$= -\{ \frac{4}{h} b_i^{5/4} (- \frac{h}{4} b_i^{-5/4} b_i' + O(h^2)) + b_i'\}$$

$$\cdot (x - x_i) + O(h^2)$$

$$= O(h^2)$$

for $x_i \leqq x \leqq x_{i+1}$ and $0<\varepsilon\leqq 1$. To get the second estimate, we set $g(x,\varepsilon)=f(x,\varepsilon)/b(x,\varepsilon)^{3/4}$. Then $F(x,\varepsilon)$ can be written as

$$F(x,\varepsilon) = B(x,\varepsilon)^{3/4}\{ (\frac{B(x,\varepsilon)}{b(x_{i+1},\varepsilon)})^{1/4} (g(x_{i+1},\varepsilon)-g(x_i,\varepsilon))$$

$$\cdot \frac{x - x_i}{h} + g(x_i,\varepsilon)\},$$

$$x_i \leqq x \leqq x_{i+1}, \quad 0<\varepsilon\leqq 1.$$

A further estimation yields

$$F(x,\varepsilon) = (b(x,\varepsilon)^{3/4}+ O(h^2))\left[(1+ O(h^2))(g(x_{i+1},\varepsilon)-g(x_i,\varepsilon))\right.$$

$$\left. \cdot \frac{x - x_i}{h} + g(x_i,\varepsilon)\right]$$

$$= b(x,\varepsilon)^{3/4}\left[(g(x_{i+1},\varepsilon)-g(x_i,\varepsilon))\frac{x - x_i}{h} + g(x_i,\varepsilon)\right]$$

$$+ O(h^2).$$

Since the bracketed term of the right hand side gives a linear interpolation to $g(x,\varepsilon)$, we finally have

$$F(x,\varepsilon) = b(x,\varepsilon)^{3/4}\left[g(x,\varepsilon) + O(h^2)\right] + O(h^2)$$

$$= f(x,\varepsilon) + O(h^2).$$

This completes the proof.

3. Construction of a difference scheme

In this section, we show by applying the Liouville-Green transformation to (2.1a) that its solution $Y(x)$ can be constructed in each subinterval $[x_i, x_{i+1}]$, and derive a three-point difference scheme between $Y_{i-1} = Y(x_{i-1})$, $Y_i = Y(x_i)$ and $Y_{i+1} = Y(x_{i+1})$.

In the subinterval $[x_i, x_{i+1}]$, we change the equation (2.1a), by using the Liouville-Green transformation

$$v_i = \psi_i(x)Y(x), \qquad z = \phi_i(x),$$

into

$$\varepsilon \frac{d^2 v_i}{dz^2} + \frac{\varepsilon}{\phi_i'^2} \left(\phi_i'' - 2\frac{\psi_i'}{\psi_i} \phi_i' \right) \frac{dv_i}{dz}$$

$$- \frac{1}{\phi_i'^2} \left\{ B(x,\varepsilon) + \varepsilon\left(\left(\frac{\psi_i'}{\psi_i} \right)' - \left(\frac{\psi_i'}{\psi_i} \right)^2 \right) \right\} v_i = \frac{\psi_i}{\phi_i'^2} F(x,\varepsilon). \tag{3.1}$$

We determine $\psi_i(x)$ and $\phi_i'(x)$ so as to satisfy the equations

$$\phi_i'' = 2 \frac{\psi_i'}{\psi_i} \phi_i'$$

and

$$\left(\frac{\psi_i'}{\psi_i} \right)' = \left(\frac{\psi_i'}{\psi_i} \right)^2$$

which can be resolved analytically. Indeed, we get

$$\psi_i(x) = -\frac{1}{d_1(x-x_i) + d_2}$$

and

$$\phi_i'(x) = \frac{1}{(d_1(x-x_i) + d_2)^2}, \qquad (3.2)$$

where d_1 and d_2 are constants. It is easily seen that the choices $d_1 = \beta_i/h$ and $d_2 = \alpha_i$ lead to the equality $\phi_i'^2(x) = B(x,\varepsilon)$ for $x_i \leqq x \leqq x_{i+1}$.

Let $\phi(x)$ be a function obtained by solving (3.2) successively and by connecting these functions each other at the nodes x_i. It is not hard to verify that

$$\phi(x) = \phi(x_i) + \frac{x - x_i}{\alpha_i(\beta_i(x-x_i)/h + \alpha_i)}$$

$$x_i \leqq x \leqq x_{i+1}.$$

This shows that $\phi(x)$ is continuously differentiable and monotonically increasing on $[0,1]$. Hence the equation $z=\phi(x)$ has an inverse $x=\phi^{-1}(z)$. Thus the equation (3.1) can be written as

$$\varepsilon \frac{d^2 v_i}{dz^2} - v_i = \frac{\psi_i(\phi^{-1}(z))}{\phi_i'(\phi^{-1}(z))^2} F(\phi^{-1}(z),\varepsilon), \qquad (3.3)$$

$$\phi(x_i) \leqq z \leqq \phi(x_{i+1}),$$

whose solution takes the form

$$v_i(z) = K_1 \exp((z-\phi(x_i))/\sqrt{\varepsilon}) + K_2 \exp(-(z-\phi(x_i))/\sqrt{\varepsilon}) + w_i(z).$$

$$(3.4)$$

Here $w_i(z)$ denotes a particular solution of (3.3). Since the explicit form of $w_i(z)$ is not required now, we carry over its calculation later. Using (3.4), the solution $Y(x)$ of (2.1a) in the subinterval $[x_i, x_{i+1}]$ is expressible in the form

$$\begin{aligned} Y(x) &= v_i(\phi(x))/\psi_i(x) \\ &= -\left(\frac{\beta_i}{h}(x-x_i)+\alpha_i\right)\{K_1 \exp((\phi(x)-\phi(x_i))/\sqrt{\varepsilon}) \\ &\qquad + K_2 \exp(-(\phi(x)-\phi(x_i))/\sqrt{\varepsilon}) + w_i(\phi(x))\}. \end{aligned} \tag{3.5}$$

To derive a three-point difference scheme between Y_{i-1}, Y_i and Y_{i+1}, we represent K_1 and K_2 by Y_i and Y_{i+1}. This is accomplished by solving a system

$$Y_i = -\alpha_i(K_1 + K_2 + w_{i,i})$$

and

$$Y_{i+1} = -\alpha_{i+1}\left(\tau_i K_1 + \frac{1}{\tau_i} K_2 + w_{i,i+1}\right),$$

where $\tau_i = \exp(\rho/\alpha_i\alpha_{i+1})$, $\rho = h/\sqrt{\varepsilon}$ and $w_{i,j} = w_i(\phi(x_j))$. The solutions K_1 and K_2 are

$$K_1 = \left(-\frac{1}{\alpha_i\tau_i} Y_i + \frac{1}{\alpha_{i+1}} Y_{i+1} + w_{i,i+1} - \frac{1}{\tau_i} w_{i,i}\right)/\left(\frac{1}{\tau_i} - \tau_i\right)$$

and

$$K_2 = \left(\frac{\tau_i}{\alpha_i} Y_i - \frac{1}{\alpha_{i+1}} Y_{i+1} - w_{i,i+1} + \tau_i w_{i,i}\right)/\left(\frac{1}{\tau_i} - \tau_i\right),$$

respectively.

In the (i-1)th interval $[x_{i-1}, x_i]$, we repeat the same discussion. That is, we set

$$Y(x) = -\left(\frac{\beta_{i-1}}{h}(x-x_{i-1})+\alpha_{i-1}\right)\{ L_1\exp((\phi(x)-\phi(x_{i-1}))/\sqrt{\varepsilon})$$

$$+ L_2\exp(-(\phi(x)-\phi(x_{i-1}))/\sqrt{\varepsilon}) + w_{i-1}(\phi(x))\}, \quad (3.6)$$

and represent L_1 and L_2 by Y_{i-1} and Y_i. We find at once that L_1 and L_2 are obtainable by replacing the index i appearing in K_1 and K_2, respectively, by i-1.

It remains only to connect the first derivative of Y(x) in (3.5) with that of Y(x) in (3.6) at the node x_i. After a careful computation, we obtain

$$-\frac{\sigma_{i-1}}{\sinh(\sigma_{i-1})} Y_{i-1} + \left\{\frac{\rho}{\alpha_i 2}(\coth(\sigma_{i-1})+\coth(\sigma_i))-\frac{\alpha_{i+1}-2\alpha_i+\alpha_{i-1}}{\alpha_i}\right\}$$

$$\cdot Y_i - \frac{\sigma_i}{\sinh(\sigma_i)} Y_{i+1} = G_i, \quad (3.7)$$

where $\sigma_i = \rho/\alpha_i\alpha_{i+1}$ and

$$G_i = \frac{\rho}{\alpha_i}\left\{\frac{1}{\sinh(\sigma_{i-1})} w_{i-1,i-1} - \coth(\sigma_{i-1})w_{i-1,i}\right.$$

$$- \coth(\sigma_i)w_{i,i} + \frac{1}{\sinh(\sigma_i)} w_{i,i+1}$$

$$\left. + \sqrt{\varepsilon}\left(\frac{dw_{i-1,i}}{dz} - \frac{dw_{i,i}}{dz} \right)\right\}.$$

To calculate G_i, we seek $w_i(z)$. Note that

$$\frac{\psi_i(\phi^{-1}(z))}{\phi_i'(\phi^{-1}(z))^2} F(\phi^{-1}(z),\varepsilon) = -\{\alpha_i\alpha_{i+1}\frac{\alpha_{i+1}^3 f_{i+1}-\alpha_i^3 f_i}{h}(z-\phi(x_i)) + \alpha_i^3 f_i\}$$

which follows from the definition of $F(x,\varepsilon)$. Since the right-hand side is a linear function of z, we easily get

$$w_i(z) = \alpha_i\alpha_{i+1}\frac{\alpha_{i+1}^3 f_{i+1}-\alpha_i^3 f_i}{h}(z-\phi(x_i)) + \alpha_i^3 f_i.$$

This makes possible the calculation of G_i. Indeed, G_i is given by

$$\begin{aligned} G_i = &-(\alpha_{i-1}^4 - \frac{\rho}{\sinh(\sigma_{i-1})}\frac{\alpha_{i-1}^3}{\alpha_i})f_{i-1} \\ &-\{\rho(\coth(\sigma_{i-1})+\coth(\sigma_i))\alpha_i^2 - (\alpha_{i-1}+\alpha_{i+1})\alpha_i^3\}f_i \\ &-(\alpha_{i+1}^4 - \frac{\rho}{\sinh(\sigma_i)}\frac{\alpha_{i+1}^3}{\alpha_i})f_{i+1}. \end{aligned}$$

Summarizing the results above, we have

Theorem. At all points of the mesh, the solution of the scheme (3.7) subject to $Y_0=a_0$ and $Y_N=a_1$ converges to that of (1.1) uniformly in ε with order h^2.

References

[1] A.M.Il'in, Differencing scheme for a differential equation with a small parameter affecting the highest derivative, Math. Notes Acad. Sci. USSR 6 (1969), 596-602.

[2] J.J.H.Miller, On the convergence, uniformly in ε, of difference scheme for a two point boundary singular perturbation problem, Proc. of Conf. on " The numerical Analysis of Singular Perturbation Problems ", Academic Press (1979), 467-474.

On a three-point difference scheme for a singular perturbation problem without a first derivative term. II

By

Koichi Niijima

1. Introduction

This paper is a continuation of our recent work [2]. In [2], we derived a three-point difference scheme for a singular perturbation problem of the type

$$\varepsilon y'' - b(x,\varepsilon)y = f(x,\varepsilon), \qquad 0 \leqq x \leqq 1, \tag{1.1a}$$

$$y(0) = a_0, \quad y(1) = a_1, \tag{1.1b}$$

where ε is a parameter satisfying $0<\varepsilon\leqq 1$. And we proved by making some assumptions on $b(x,\varepsilon)$ and $f(x,\varepsilon)$ that the solution of this scheme converges to that of (1.1) uniformly in ε with order h^2, where h denotes a mesh step. In deriving such a scheme, we utilized the Liouville-Green transformation which also played an important role in the proof of the uniform convergence.

In the present paper, a three-point difference scheme, whose solution converges to that of (1.1) uniformly in ε with order h^3, is derived under some conditions slightly stronger than those in [2], when ε satisfies $0<\varepsilon\leqq\varepsilon_0$ for small ε_0. The derivation and the proof of the uniform convergence will be done by the use of another Liouville-Green transformation containing three free parameters to be determined. This transformation can be obtained by solving some differential equations analytically. But since the resolution is not so easy, we shall state its process in Appendix.

In final section, several numerical experiments are performed with the schemes of Miller [1] and of Niijima [2] as well as a new one, and the accuracy of the computed solutions is compared each other.

2. Approximation to the problem (1.1)

The functions $b(x,\varepsilon)$ and $f(x,\varepsilon)$ are assumed to be three times continuously differentiable with respect to x on $D=\{(x,\varepsilon)\,|\,0\leqq x\leqq 1,\ 0<\varepsilon\leqq\varepsilon_0\}$, and to be bounded there together with their derivatives with respect to x up to third order. Moreover, we assume that $b(x,\varepsilon)\geqq\delta>0$ in $\overline{D}$.

Let N be a positive integer and define a mesh step h by $h=1/N$. And we denote equidistant mesh points by $x_i=ih$, $i=0,\cdots,N$. In the subinterval $[x_i,x_{i+1}]$, the functions $b(x,\varepsilon)$ and $f(x,\varepsilon)$ are approximated now by

$$B(x,\varepsilon) = 1/(\alpha_i(x-x_{i+1/2})^2+\beta_i(x-x_{i+1/2})+\gamma_i)^2$$

and

$$F(x,\varepsilon) = B(x,\varepsilon)^{3/4}\{p_i(\phi(x)-\phi(x_{i+1/2}))^2 + q_i(\phi(x)-\phi(x_{i+1/2}))+r_i\}, \qquad (2.1)$$

respectively. Here α_i, β_i and γ_i denote, respectively,

$$\alpha_i = 2(d_i - 2d_{i+1/2} + d_{i+1})/h^2$$

$$\beta_i = (d_{i+1} - d_i)/h$$

and

$$\gamma_i = d_{i+1/2},$$

in which $d_i = 1/\sqrt{b(x_i,\varepsilon)}$ and $x_{i+1/2} = x_i + h/2$. Also, $\phi(x)$ is a function defined by

$$\phi(x) = \phi(x_{i+1/2}) + \int_{x_{i+1/2}}^{x} \sqrt{B(t,\varepsilon)}\,dt \tag{2.2}$$

for $x_i \le x \le x_{i+1}$, and p_i, q_i and r_i are

$$p_i = ((g_{i+1} - g_{i+1/2})k_1^{(i)} + (g_{i+1/2} - g_i)k_2^{(i)})/k_1^{(i)}k_2^{(i)}(k_2^{(i)} - k_1^{(i)}),$$

$$q_i = -((g_{i+1} - g_{i+1/2})k_1^{(i)2} + (g_{i+1/2} - g_i)k_2^{(i)2})$$

$$\div\, k_1^{(i)}k_2^{(i)}(k_2^{(i)} - k_1^{(i)})$$

and

$$r_i = g_{i+1/2},$$

respectively, where $g_i = b(x_i,\varepsilon)^{-3/4} f(x_i,\varepsilon)$, $k_1^{(i)} = \phi(x_i) - \phi(x_{i+1/2})$ and $k_2^{(i)} = \phi(x_{i+1}) - \phi(x_{i+1/2})$. For later use, we note that the function $z = \phi(x)$ has an inverse $x = \phi^{-1}(z)$ because the former is continuous and monotonically increasing on $[0,1]$. The functions $B(x,\varepsilon)$ and $F(x,\varepsilon)$ defined above approximate to $b(x,\varepsilon)$ and $f(x,\varepsilon)$, respectively, with order h^3 uniformly in ε.

This will be proved in next lemma as a preparation for establishing the estimate

$$\max_{0\leq x\leq 1} |Y(x) - y(x)| \leq Ch^3, \qquad (2.3)$$

where $Y(x)$ is a solution of

$$\varepsilon Y'' - B(x,\varepsilon)Y = F(x,\varepsilon), \qquad 0\leq x\leq 1, \qquad (2.4a)$$

$$Y(0) = a_0, \quad Y(1) = a_1, \qquad (2.4b)$$

and C is a constant independent of h and ε.

Lemma. The estimate (2.3) holds.

Proof. We find from the definition of $B(x,\varepsilon)$ that the quadratic function $\alpha_i(x-x_{i+1/2})^2+\beta_i(x-x_{i+1/2})+\gamma_i$ interpolates to $1/\sqrt{b(x,\varepsilon)}$ at the nodes x_i, $x_{i+1/2}$ and x_{i+1}. Therefore,

$$\alpha_i(x-x_{i+1/2})^2+\beta_i(x-x_{i+1/2})+\gamma_i = \frac{1}{\sqrt{b(x,\varepsilon)}} + O(h^3).$$

We thus have

$$B(x,\varepsilon) = b(x,\varepsilon) + O(h^3). \qquad (2.5)$$

In the next, we show that

$$F(x,\varepsilon) = f(x,\varepsilon) + O(h^3).$$

We first have, by virtue of (2.5),

$$F(x,\varepsilon) = b(x,\varepsilon)^{3/4}\{p_i(\phi(x)-\phi(x_{i+1/2}))^2+ q_i(\phi(x)-\phi(x_{i+1/2})) + r_i\} + O(h^3).$$

Applying the Taylor's theorem to the expression in the braces, we obtain

$$b(x,\varepsilon)^{-3/4}F(x,\varepsilon) = r_i + q_i\phi'(x_{i+1/2})(x-x_{i+1/2})$$

$$+ \frac{1}{2}(q_i\phi''(x_{i+1/2})+2p_i\phi'^2(x_{i+1/2}))(x-x_{i+1/2})^2$$

$$+ O(h^3)$$

$$= r_i + \frac{q_i}{\gamma_i}(x-x_{i+1/2}) + \frac{2p_i-\beta_i q_i}{2\gamma_i^2}(x-x_{i+1/2})^2$$

$$+ O(h^3). \qquad (2.6)$$

Since the same theorem leads to

$$k_1^{(i)} = -\frac{h}{2\gamma_i} - \frac{\beta_i}{8\gamma_i^2}h^2 + O(h^3)$$

and

$$k_2^{(i)} = \frac{h}{2\gamma_i} - \frac{\beta_i}{8\gamma_i^2}h^2 + O(h^3),$$

we have

$$p_i = \{(\frac{g'_{i+1/2}}{2}h+\frac{g''_{i+1/2}}{8}h^2+O(h^3))(-\frac{h}{2\gamma_i} - \frac{\beta_i}{8\gamma_i^2}h^2 + O(h^3))$$

$$+ (\frac{g'_{i+1/2}}{2}h-\frac{g''_{i+1/2}}{8}h^2+O(h^3))(\frac{h}{2\gamma_i} - \frac{\beta_i}{8\gamma_i^2}h^2 + O(h^3))\}$$

$$\div\{(-\frac{h}{2\gamma_i} - \frac{\beta_i}{8\gamma_i^2}h^2+O(h^3))(\frac{h}{2\gamma_i} - \frac{\beta_i}{8\gamma_i^2}h^2+O(h^3))(\frac{h}{\gamma_i} + O(h^3))\}$$

$$= \frac{g'_{i+1/2}\beta_i\gamma_i}{2} + \frac{g''_{i+1/2}\gamma_i^2}{2} + O(h).$$

Similarly, we have

$$q_i = g'_{i+1/2}\gamma_i + O(h^2).$$

Substituting these p_i and q_i into (2.6) yields

$$b(x,\varepsilon)^{-3/4}F(x,\varepsilon) = g_{i+1/2} + g'_{i+1/2}(x-x_{i+1/2}) + \frac{g''_{i+1/2}}{2}(x-x_{i+1/2})^2 + O(h^3)$$

$$= g(x,\varepsilon) + O(h^3).$$

From the definition of $g(x,\varepsilon)$, we finally get

$$F(x,\varepsilon) = f(x,\varepsilon) + O(h^3).$$

The proof of (2.3) can be done in the same way as in Lemma of [2].

3. Construction of a difference scheme

The definition of $B(x,\varepsilon)$ and $F(x,\varepsilon)$ shows that these functions are continuous on D, and that there exists a constant δ' such that $B(x,\varepsilon) \geq \delta' > 0$ in $\overline{D}$. Therefore, the problem (2.4) has a unique solution $Y(x)$ which is twice continuously differentiable on [0,1].

Now, if we apply the Liouville-Green transformation

$$v_i = \psi_i(x)Y(x), \qquad z = \phi_i(x) \tag{3.1}$$

to (2.4a) in i th interval $[x_i, x_{i+1}]$, then (2.4a) changes into

$$\varepsilon \frac{d^2 v_i}{dz^2} + \frac{\varepsilon}{\phi_i'^2}\left(\phi_i'' - 2\frac{\psi_i'}{\psi_i}\phi_i'\right)\frac{dv_i}{dz}$$

$$- \frac{1}{\phi_i'^2}\{B(x,\varepsilon) + \varepsilon((\frac{\psi_i'}{\psi_i})' - (\frac{\psi_i'}{\psi_i})^2)\}v_i = \frac{\psi_i}{\phi_i'^2} F(x,\varepsilon). \tag{3.2}$$

We determined, in [2], ϕ_i' and ψ_i so that $\phi_i'' - 2\frac{\psi_i'}{\psi_i}\phi_i'$ and $(\frac{\psi_i'}{\psi_i})' - (\frac{\psi_i'}{\psi_i})^2$ vanish, but now we let

$$\phi_i'' - 2\frac{\psi_i'}{\psi_i}\phi_i' = k_i \phi_i'^2 \tag{3.3}$$

and

$$(\frac{\psi_i'}{\psi_i})' - (\frac{\psi_i'}{\psi_i})^2 = \ell_i \phi_i'^2, \tag{3.4}$$

where k_i and ℓ_i are constants. By solving (3.3) and (3.4), we have

$$\phi_i'(x) = 1/(c_1(x-x_{i+1/2})^2 + c_2(x-x_{i+1/2}) + c_3) \tag{3.5}$$

and

$$\frac{\psi_i'(x)}{\psi_i(x)} = \frac{c_1(x-x_{i+1/2}) + (c_2-k_i)/2}{c_1(x-x_{i+1/2})^2 + c_2(x-x_{i+1/2}) + c_3}, \tag{3.6}$$

where $\ell_i = \frac{(c_2+k_i)c_2}{2} - (\frac{c_2+k_i}{2})^2 - c_1 c_3$ (See Appendix).

As is easily seen from Appendix, the freedom of k_i does not contribute to increasing free parameters in $\phi_i'(x)$. Hence we let $k_i = 0$ for brevity. Then, by solving (3.6), we obtain

$$\psi_i(x) = 1/(c_1(x-x_{i+1/2})^2+c_2(x-x_{i+1/2})+c_3)^{1/2}.$$

Now, we choose as $c_1 = \alpha_i$, $c_2 = \beta_i$ and $c_3 = \gamma_i$. Such choices change (3.2) into

$$\varepsilon \frac{d^2 v_i}{dz^2} - (1 + \varepsilon \ell_i) v_i = B(x,\varepsilon)^{-3/4} F(x,\varepsilon), \qquad (3.7)$$

where $\ell_i = \beta_i^2/4 - \alpha_i \gamma_i$. Notice here that $\phi(x)$ in (2.2) is just obtained by integrating (3.5) from $x_{i+1/2}$ to x and by connecting them each other. Since $z=\phi(x)$ has an inverse $x=\phi^{-1}(z)$, the right hand side of (3.7) may be written as $B(\phi^{-1}(z),\varepsilon)^{-3/4}F(\phi^{-1}(z),\varepsilon)$. Thus the equation (3.7) is solvable analytically on $[\phi(x_i),\phi(x_{i+1})]$ and its solution takes the form

$$v_i(z) = K_1 \exp(s_i(z-\phi(x_{i+1/2}))) + K_2 \exp(-s_i(z-\phi(x_{i+1/2}))) + w_i(z),$$

where $s_i = \sqrt{\dfrac{1+\varepsilon \ell_i}{\varepsilon}}$, and $w_i(z)$ denotes a particular solution of (3.7). According to (3.1), we further have, for $x_i \leq x \leq x_{i+1}$,

$$\begin{aligned} Y(x) = \frac{v_i(\phi(x))}{\psi_i(x)} &= \sqrt{\alpha_i(x-x_{i+1/2})^2+\beta_i(x-x_{i+1/2})+\gamma_i} \\ &\cdot \{K_1 \exp(s_i(\phi(x)-\phi(x_{i+1/2}))) \\ &+ K_2 \exp(-s_i(\phi(x)-\phi(x_{i+1/2}))) + w_i(\phi(x))\}. \end{aligned} \qquad (3.8)$$

The discussion below is exactly the same as in [2]. We represent K_1 and K_2 in (3.8) by $Y_i = Y(x_i)$ and $Y_{i+1} = Y(x_{i+1})$ to obtain

$$K_1 = -\left(\frac{b_i^{1/4}}{\tau_{i,2}} Y_i - \frac{b_{i+1}^{1/4}}{\tau_{i,1}} Y_{i+1} - \frac{w_{i,i}}{\tau_{i,2}} + \frac{w_{i,i+1}}{\tau_{i,1}} \right) \div \left(\frac{\tau_{i,2}}{\tau_{i,1}} - \frac{\tau_{i,1}}{\tau_{i,2}} \right)$$

and

$$K_2 = \left(b_i^{1/4}\tau_{i,2}Y_i - b_{i+1}^{1/4}\tau_{i,1}Y_{i+1} - \tau_{i,2}w_{i,i} + \tau_{i,1}w_{i,i+1} \right) \div \left(\frac{\tau_{i,2}}{\tau_{i,1}} - \frac{\tau_{i,1}}{\tau_{i,2}} \right),$$

where $\tau_{i,j} = \exp(s_i k_j^{(i)})$ for $j=1,2$, and $w_{i,j} = w_i(\phi(x_j))$ for $j=i,i+1$. We repeat the same procedure as above in $(i-1)$th interval $[x_{i-1}, x_i]$. And we connect the first derivative of $Y(x)$ in i th interval with that of $Y(x)$ in $(i-1)$th interval at the node x_i. By doing so, we obtain the difference scheme

$$\frac{s_{i-1}}{\sinh(\sigma_{i-1})} b_{i-1}^{1/4} Y_{i-1} - \{(d_{i-1} - 4d_{i-1/2} + 6d_i - 4d_{i+1/2} + d_{i+1})/2h$$
$$+ \coth(\sigma_{i-1}) s_{i-1} + \coth(\sigma_i) s_i\} b_i^{1/4} Y_i + \frac{s_i}{\sinh(\sigma_i)} b_{i+1}^{1/4} Y_{i+1}$$
$$= G_i, \qquad (3.9)$$

where $\sigma_i = s_i(k_2^{(i)} - k_1^{(i)})$ and

$$G_i = \frac{s_{i-1}}{\sinh(\sigma_{i-1})} w_{i-1,i-1} - \coth(\sigma_{i-1}) s_{i-1} w_{i-1,i}$$

$$- \coth(\sigma_i) s_i w_{i,i} + \frac{s_i}{\sinh(\sigma_i)} w_{i,i+1}$$

$$-(2p_{i-1}k_2^{(i-1)} + q_{i-1})/s_{i-1}^2\varepsilon + (2p_i k_1^{(i)} + q_i)/s_i^2\varepsilon \ .$$

Finally, we calculate G_i. Since we have, from (2.1),

$$B(\phi^{-1}(z),\varepsilon)^{-3/4}F(\phi^{-1}(z),\varepsilon) = p_i(z-\phi(x_{i+1/2}))^2 + q_i(z-\phi(x_{i+1/2})) + r_i,$$

the particular solution $w_i(z)$ may be written as

$$w_i(z) = -\{p_i(z-\phi(x_{i+1/2}))^2 + q_i(z-\phi(x_{i+1/2})) + (\frac{2p_i}{s_i^2} + r_i)\}/s_i^2\varepsilon$$

$$= -(B(\phi^{-1}(z),\varepsilon)^{-3/4}F(\phi^{-1}(z),\varepsilon) + \frac{2p_i}{s_i^2})/s_i^2\varepsilon.$$

Therefore,

$$G_i = -\frac{b_{i-1}^{-3/4} f_{i-1}}{s_{i-1}\varepsilon\sinh(\sigma_{i-1})} + (\frac{\coth(\sigma_{i-1})}{s_{i-1}\varepsilon} + \frac{\coth(\sigma_i)}{s_i\varepsilon}) b_i^{-3/4} f_i$$

$$- \frac{b_{i+1}^{-3/4} f_{i+1}}{s_i\varepsilon\sinh(\sigma_i)} + \frac{2p_{i-1}}{s_{i-1}^3\varepsilon}(\coth(\sigma_{i-1}) - \frac{1}{\sinh(\sigma_{i-1})})$$

$$+ \frac{2p_i}{s_i^3\varepsilon}(\coth(\sigma_i) - \frac{1}{\sinh(\sigma_i)}) - (2p_{i-1}k_2^{(i-1)} + q_{i-1})/s_{i-1}^2\varepsilon$$

$$+ (2p_i k_1^{(i)} + q_i)/s_i^2\varepsilon \ .$$

Concerning the difference scheme (3.9), we have the following theorem.

Theorem. At all points of the mesh, the solution of the scheme (3.9) subject to $Y_0=a_0$ and $Y_N=a_1$ converges to that of (1.1) uniformly in ε with order h^3.

Proof. The proof follows directly from Lemma.

4. Numerical experiments

We first rewrite the difference scheme (3.9) in the following form;

$$2\nu_{i-1}b_{i-1}^{1/4}Y_{i-1}-\{(d_{i-1}-4d_{i-1/2}+6d_i-4d_{i+1/2}+d_{i+1})\frac{1-\nu_{i-1}^2}{2hs_{i-1}}+\eta_i\}$$

$$\cdot b_i^{1/4}Y_i+\frac{2\nu_i}{\lambda_i}b_{i+1}^{1/4}Y_{i+1}$$

$$=-\frac{2\nu_{i-1}}{s_{i-1}^2\varepsilon}b_{i-1}^{-3/4}f_{i-1}+(\frac{1+\nu_{i-1}^2}{s_{i-1}^2\varepsilon}+\frac{1+\nu_i^2}{s_i^2\varepsilon\lambda_i})b_i^{-3/4}f_i-\frac{2\nu_i}{s_i^2\varepsilon\lambda_i}b_{i+1}^{-3/4}f_{i+1}$$

$$+\frac{2p_{i-1}}{s_{i-1}^4\varepsilon}(1-\nu_{i-1})^2+\frac{2p_i}{s_i^4\varepsilon\lambda_i}(1-\nu_i)^2+\{\frac{2p_ik_1^{(i)}+q_i}{s_i^2\varepsilon}-\frac{2p_{i-1}k_2^{(i-1)}+q_{i-1}}{s_{i-1}^2\varepsilon}\}$$

$$\cdot\frac{1-\nu_{i-1}^2}{s_{i-1}}, \qquad (4.1)$$

where $\nu_i=\exp(-\sigma_i)$, $\lambda_i=s_{i-1}(1-\nu_i^2)/s_i(1-\nu_{i-1}^2)$ and

$$\eta_i=1+\nu_{i-1}^2+\frac{1-\nu_{i-1}^2}{1-\nu_i^2}(1+\nu_i^2)\frac{s_i}{s_{i-1}}.$$

This is because we want to prevent numerical overflows due to exponential growth. In each table below, we give some results for the scheme (4.1), and the schemes of Miller [1] and of Niijima [2]. For simplicity, only the maximum error at the nodes is listed in each table. The experiments were carried out for N=8, 16 and 32.

The first problem is

$$\varepsilon y'' - (x+1)y = 40(x(x^2-1)-2\varepsilon),$$
$$y(0) = y(1) = 0$$

with the exact solution $y(x)=40x(1-x)$.

N=8

	Scheme (4.1)	Miller [1]	Niijima [2]
$\varepsilon = 10^{-2}$	0.20 (-4)	0.93 (-1)	0.10 (0)
10^{-3}	0.24 (-4)	0.51 (-1)	0.81 (-1)
10^{-4}	0.20 (-4)	0.71 (-2)	0.40 (-1)
10^{-5}	0.21 (-4)	0.71 (-3)	0.14 (-1)
10^{-6}	0.19 (-4)	0.69 (-4)	0.46 (-2)
10^{-7}	0.19 (-4)	0.57 (-5)	0.15 (-2)
10^{-8}	0.67 (-5)	0.0	0.47 (-3)

N=16

	Scheme (4.1)	Miller [1]	Niijima [2]
$\varepsilon = 10^{-2}$	0.65 (-4)	0.25 (-1)	0.26 (-1)
10^{-3}	0.36 (-4)	0.21 (-1)	0.24 (-1)
10^{-4}	0.33 (-4)	0.70 (-2)	0.17 (-1)
10^{-5}	0.26 (-4)	0.75 (-3)	0.69 (-2)
10^{-6}	0.20 (-4)	0.74 (-4)	0.23 (-2)
10^{-7}	0.20 (-4)	0.67 (-5)	0.76 (-3)
10^{-8}	0.11 (-4)	0.0	0.24 (-3)

N=32

	Scheme (4.1)	Miller [1]	Niijima [2]
$\varepsilon = 10^{-2}$	0.59 (-4)	0.64 (-2)	0.65 (-2)
10^{-3}	0.21 (-4)	0.61 (-2)	0.64 (-2)
10^{-4}	0.35 (-4)	0.41 (-2)	0.56 (-2)
10^{-5}	0.39 (-4)	0.77 (-3)	0.31 (-2)
10^{-6}	0.21 (-4)	0.76 (-4)	0.12 (-2)
10^{-7}	0.21 (-4)	0.67 (-5)	0.38 (-3)
10^{-8}	0.14 (-4)	0.0	0.13 (-3)

The next problem is

$$\varepsilon y'' - \frac{4}{(x+1)^4}(1+\sqrt{\varepsilon}(x+1))y = \frac{4}{(x+1)^4}\{((1+\sqrt{\varepsilon}(x+1))+4\pi^2\varepsilon)\cos(2\pi t)$$
$$-2\pi\varepsilon(x+1)\sin(2\pi t) + \frac{3(1+\sqrt{\varepsilon}(x+1))\xi(1)}{1-\xi(1)}\},$$

$$y(0) = 2, \qquad y(1) = -1$$

with the exact solution $y(x) = -\cos(2\pi t) + \frac{3(\xi(t)-\xi(1))}{1-\xi(1)}$, where $\xi(t) = \exp(-t/\sqrt{\varepsilon})$ and $t=2x/(x+1)$.

N=8

	Scheme (4.1)	Miller [1]	Niijima [2]
$\varepsilon = 10^{-2}$	0.21 (-2)	0.91 (-1)	0.64 (-1)
10^{-3}	0.24 (-2)	0.30 (-1)	0.50 (-1)
10^{-4}	0.17 (-2)	0.39 (-2)	0.22 (-1)
10^{-5}	0.67 (-3)	0.41 (-3)	0.78 (-2)
10^{-6}	0.48 (-3)	0.40 (-4)	0.26 (-2)
10^{-7}	0.73 (-4)	0.39 (-5)	0.82 (-3)
10^{-8}	0.24 (-4)	0.36 (-6)	0.26 (-3)

N=16

	Scheme (4.1)	Miller [1]	Niijima [2]
$\varepsilon = 10^{-2}$	0.11 (-3)	0.25 (-1)	0.18 (-1)
10^{-3}	0.17 (-3)	0.15 (-1)	0.21 (-1)
10^{-4}	0.20 (-3)	0.38 (-2)	0.11 (-1)
10^{-5}	0.98 (-4)	0.41 (-3)	0.42 (-2)
10^{-6}	0.32 (-4)	0.40 (-4)	0.14 (-2)
10^{-7}	0.11 (-4)	0.39 (-5)	0.45 (-3)
10^{-8}	0.40 (-5)	0.36 (-6)	0.14 (-3)

N=32

	Scheme (4.1)	Miller [1]	Niijima [2]
$\varepsilon = 10^{-2}$	0.62 (-4)	0.65 (-2)	0.48 (-2)
10^{-3}	0.26 (-4)	0.71 (-2)	0.77 (-2)
10^{-4}	0.28 (-4)	0.26 (-2)	0.66 (-2)
10^{-5}	0.27 (-4)	0.40 (-3)	0.28 (-2)
10^{-6}	0.86 (-5)	0.41 (-4)	0.97 (-3)
10^{-7}	0.32 (-5)	0.39 (-5)	0.31 (-3)
10^{-8}	0.30 (-5)	0.66 (-6)	0.10 (-3)

When ε is small enough, the scheme of Miller [1] is superior to the others. But the new scheme (4.1) gives good results in the case when ε is not so small.

Appendix

We shall solve (3.3) and (3.4). For brevity, we omit the index i. Substituting $\psi'/\psi=(\phi''-k\phi'^2)/2\phi'$ which follows from (3.3) into (3.4) yields

$$2\phi'\phi''' - 3\phi''^2 = m\phi'^4, \tag{A.1}$$

where $m=4\ell+k^2$. We set $\mu=\phi'$, and further

$$\omega = \mu'/\mu^2. \tag{A.2}$$

Then (A.1) changes into

$$\frac{2\omega'}{m-\omega^2} = \mu. \tag{A.3}$$

Since $\omega'=\frac{d\omega}{d\mu}\mu' = \frac{d\omega}{d\mu}\mu^2\omega$ from (A.2), the equation (A.3) may be written as

$$\frac{2\omega\, d\omega}{m-\omega^2} = \frac{d\mu}{\mu}.$$

By integrating the both sides, we obtain

$$\omega^2 = m - \frac{c_4}{\mu}, \tag{A.4}$$

where c_4 is non-zero. Combining (A.4) with (A.2), we have

$$-\frac{d\mu}{\mu^{3/2}\sqrt{m\mu - c_4}} = dx.$$

Since

$$\frac{d}{d\mu}\left(\frac{2\sqrt{m\mu - c_4}}{c_4\mu^{1/2}}\right) = \frac{1}{\mu^{3/2}\sqrt{m\mu - c_4}},$$

it follows that for a constant c_5,

$$\frac{2\sqrt{m\mu - c_4}}{c_4 \mu^{1/2}} = x + c_5 .$$

Solving this equation, we get

$$\mu(x) = 1/(-\frac{c_4}{4}(x+c_5)^2 + \frac{m}{c_4}) . \qquad \text{(A.5)}$$

Here we set $c_1 = -c_4/4$, $c_2 = -c_4 c_5/2$ and $c_3 = m/c_4 - c_4 c_5{}^2/4$. Since c_4 is non-zero, we can express c_4, c_5 and m by c_1, c_2 and c_3. That is, $\mu(x)$ in (A.5) may be written as

$$\phi'(x) = \mu(x) = 1/(c_1 x^2 + c_2 x + c_3)$$

with free parameters c_1, c_2 and c_3.

References

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On Von Foerster Equation in Biomathematics

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Chapter 1. Introduction

It is well known that the aspect of growth of microorganisms in a finite amount of liquid medium (batch culture) changes from "logarithmic phase" (period of exponential growth) to "stationary phase" (period of no growth) by action of many limiting factors such as deficiency of nutrients, accumulation of harmful metabolites, shift of pH and so on. According to the experiment by Maruyama et al. [1] on *Bacillus subtilis* with glucose deficiency as the only limiting factor, the total mass of the cells increases exponentially till the starvation point (t_s) at which glucose is almost exhausted, and the increase abruptly stops at t_s . On the other hand, the growth of the total number of the cells begins to slow down about 1.5 generation time before t_s (Generation time is the time length between two divisions of a cell occurring in succession.), showing that the culture has entered into transition period from logarithmic period. Nishi et al. [2] suggested that synchronization of cell cycle occurs during this period.

Starting from a few simple assumptions, we shall mathematically show that these phenomena in the transition phase can be explained by the model in which generation time of the cells gets longer when concentration of a limiting nutrient becomes lower than a certain critical value.

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Chapter 2. Theory

§ 1. Preliminaries

Letting a be cell age of a cell (time having elapsed since the last division of that cell) and $u(a, t)$ be the density function of cell number with respect to a at time t, we have

$$(2.1) \quad \frac{\partial u}{\partial a} + \frac{\partial u}{\partial t} = -\lambda u \qquad \text{(Von Foerster Equation [3])},$$

where λ is called loss function and generally depends on a, t and u. In our case, λ represents death rate of the cells since in batch culture there is no cell to be lost by emigration. In application there is a case in which $u(a, t)$ is not necessarily of class C^1. In such a case we may consider from the biological point of view that $u(a, t)$ is differentiable at least in the direction of characteristic line of Von Foerster equation (2.1). We consider, therefore, the derivative in the direction of characteristic line of (2.1):

$$D_c u(a, t) = \lim_{h \to 0} \frac{u(a + h, t + h) - u(a, t)}{h}$$

which naturally coincides with $(\partial/\partial a + \partial/\partial t)u(a, t)$ when u is of class C^1. Hence we treat Von Foerster equation in the form:

$$(F) \quad D_c u(a, t) = -\lambda u(a, t) .$$

Assumptions:

(1) Generation time a_g is a function of the concentration C of the limiting nutrient (glucose). (As C is a function of t, we consider a_g a function of t in the following treatment.)

(2) All the cells have the same generation time.

(3) A cell whose cell age has reached generation time (i.e. $a = a_g(t)$)

divides into two equal sister cells.

We suppose that $a_g(t)$ is a positive, continuous and right differentiable function on $[t_0, \infty)$ and satisfies

(2.2) $\quad t < t' \implies a_g(t') - a_g(t) < t' - t \quad .$

If $a_g(t') - a_g(t) \geqslant t' - t$ for some $t < t'$, there exists a subinterval $[\tau, \tau']$ of $[t, t']$ such that $a_g(\tau') - a_g(\tau) \geqslant \tau' - \tau > 0$ and that $\tau' - \tau$ is sufficiently small. Since there is no cell division during the time interval $[\tau, \tau']$, it belongs to a period of no growth which we do not treat in the present note. The domain of definition of u is

$$D = \{(a, t) \mid t \geqslant t_0 \ , \ 0 \leqslant a \leqslant a_g(t)\} \ ,$$

where t_0 is the initial time. By assumption (2.2), D satisfies the characteristic line condition [4]:

If $(a, t) \in D$ and $(a + h, t + h) \in D$, then $(a + \theta h, t + \theta h) \in D$ for all $\theta \in [0, 1]$.

This condition is natural from the biological point of view, because $(a + \theta h, t + \theta h)$ lies on the growth trajectory in the (a, t)-plane of a cell of age a at time t . From assumption (3),

(2.3) $$\int_0^h u(a, t + h)da = 2\int_{a_g(t + h) - h}^{a_g(t)} u(a, t)da \quad (0 < h < a_g(t + h)) \ ,$$

because both sides of this equation represent the number of the cells born during the time interval $[t, t + h]$. When $u(a, t)$ is left continuous with respect to a at $(a_g(t), t)$ and $u(a,t)$ restricted on domain $\{(a, \tau) \in D \mid 0 \leqslant a \leqslant \tau - t, t \leqslant \tau < t + \varepsilon\}$ for some $\varepsilon > 0$ is continuous at $(0, t)$, dividing both sides of (2.3) by h and letting $h \to 0$, we obtain a boundary condition :

(B) $\quad u(0, t) = 2(1 - D_+a_g(t))u(a_g(t), t) \ .$

When we put $b(t) = t - a_g(t)$, b is a continuous and right differentiable function on $[t_0, \infty)$. $b(t)$ represents the birth time of a cell dividing at time t. Since b is strictly increasing from assumption (2.2), we can inductively define a sequence $t_0 < t_1 < t_2 < \cdots$ by recurrence formula :

$$t_n = b^{-1}(t_{n-1}) \qquad (n \geq 1) .$$

When $t_{n-1} \geq \lim_{t\to\infty} b(t)$ for some n, we define $t_n = \infty$ and the sequence terminates at n. Otherwise $\{t_n\}$ is an infinite sequence. For convenience sake we set $t_{-1} = b(t_0)$. When we define

$$\begin{cases} D_0 = \{(a, t) \mid t \geq t_0,\ t - a < t_0,\ 0 \leq a \leq a_g(t)\} \\ D_n = \{(a, t) \mid t_{n-1} \leq t - a < t_n,\ 0 \leq a \leq a_g(t)\} \qquad (n \geq 1) , \end{cases}$$

domain D can be expressed as the disjoint union of subdomain D_n's (Figure 1) :

(2.4) $$D = \bigcup_{n\geq 0} D_n .$$

Define a sequence of functions b_n on $[t_{n-1}, \infty)$ by recurrence formula :

$$\begin{cases} b_0(t) = t \\ b_n(t) = b_{n-1}(b(t)) \qquad (n \geq 1) . \end{cases}$$

By this definition it is clear that

$$b_1(t) = b(t) \qquad (t \geq t_0)$$

(2.5) $$b_n(t_{n-1}) = b_{n-1}(t_{n-2}) = \cdots = b_1(t_0) = b_0(t_{-1}) = t_{-1}$$

(2.6) $$b_n(t_n) = b_{n-1}(t_{n-1}) = \cdots = b_1(t_1) = b_0(t_0) = t_0 .$$

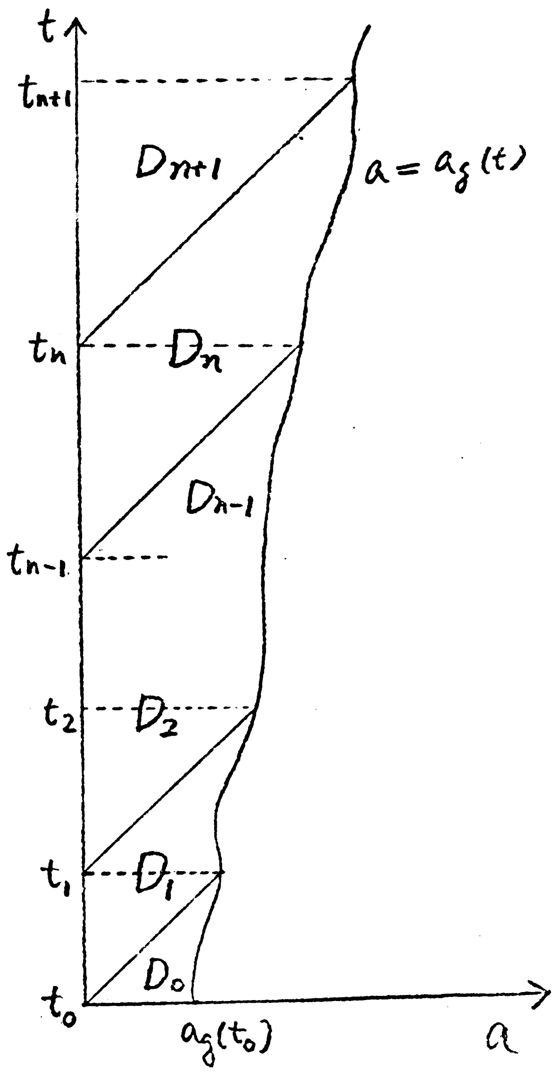

Figure 1.
Division of domain D into subdomain D_n's.

§ 2. Density Function of Cell Number

Under the notation and assumptions in § 1, the following theorem holds concerning density function $u(a, t)$ of cell number.

Theorem 1.

Let λ be a constant and φ a real-valued function defined on $(0, a_g(t_0)]$. Then, there exists a unique solution $u(a, t)$ on D of the equation

$$(2.7)\quad \begin{cases} (F) & D_c u(a, t) = -\lambda u(a, t) \\ (I) & u(a, t_0) = \varphi(a) \qquad (0 < a \leqslant a_g(t_0)) \\ (B) & u(0, t) = 2(1 - D_+ a_g(t)) u(a_g(t), t) \qquad (t \geqslant t_0) \,. \end{cases}$$

The solution $u(a, t)$ is given by the following formula :

$$(2.8)\quad u(a, t) = 2^n D_+ b_n(t - a)\, \varphi(t_0 - b_n(t - a)) e^{-\lambda(t - t_0)} \quad \text{for } (a, t) \in D_n \,.$$

Proof

(Existence) We shall show that $u(a, t)$ given by formula (2.8) is a solution of equation (2.7). When $(a, t) \in D_n$, we have $t_{n-1} \leqslant t - a < t_n$, therefore $b_n(t - a)$ makes sense. The relation

$$t_{-1} = b_n(t_{n-1}) \leqslant b_n(t - a) < b_n(t_n) = t_0$$

implies

$$0 < t_0 - b_n(t - a) \leqslant t_0 - t_{-1} = a_g(t_0) \,,$$

which shows that $t_0 - b_n(t - a)$ belongs to the domain of definition of φ. Since D is the disjoint union of D_n's $(n \geqslant 0)$ by (2.4), $u(a, t)$ is a well-defined function on D. If $(a, t) \in D_n$ and $(a + h, t + h) \in D$, then we have $(a + h, t + h) \in D_n$ and from (2.8),

$$\begin{aligned} & u(a + h, t + h) - u(a, t) \\ & = 2^n D_+ b_n(t - a)\, \varphi(t_0 - b_n(t - a)) \{ e^{-\lambda(t+h-t_0)} - e^{-\lambda(t - t_0)} \} \,. \end{aligned}$$

Hence, $u(a, t)$ satisfies Von Foerster equation (F). For $0 < a \leqslant a_g(t_0)$, we have $(a, t_0) \in D_0$, and so

$$u(a, t_0) = \varphi(t_0 - (t_0 - a))e^{-\lambda(t_0-t_0)} = \varphi(a),$$

which is no other than initial condition (I). For $t \geqslant t_0$, there exists a unique integer $n \geqslant 1$ such that $t_{n-1} \leqslant t < t_n$. Since $(0, t) \in D_n$, we have, on the one hand,

$$u(0, t) = 2^n D_+ b_n(t) \varphi(t_0 - b_n(t))e^{-\lambda(t-t_0)}.$$

On the other hand, from the relation

$$t_{n-2} = b(t_{n-1}) \leqslant b(t) < b(t_n) = t_{n-1},$$

we have $(a_g(t), t) \in D_{n-1}$, and so

$$\begin{aligned} u(a_g(t), t) &= 2^{n-1} D_+ b_{n-1}(t - a_g(t)) \varphi(t_0 - b_{n-1}(t - a_g(t)))e^{-\lambda(t-t_0)} \\ &= 2^{n-1} D_+ b_{n-1}(b(t)) \varphi(t_0 - b_{n-1}(b(t)))e^{-\lambda(t-t_0)}. \end{aligned}$$

Hence, we obtain

$$\begin{aligned} &2(1 - D_+ a_g(t))u(a_g(t), t) \\ &= 2D_+ b(t) \times 2^{n-1} D_+ b_{n-1}(b(t)) \varphi(t_0 - b_{n-1}(b(t)))e^{-\lambda(t-t_0)} \\ &= 2^n D_+ b_n(t) \varphi(t_0 - b_n(t))e^{-\lambda(t-t_0)}. \end{aligned}$$

Boundary condition (B) is therefore verified.

(Uniqueness) We shall show by mathematical induction on n that any solution $u(a, t)$ on D of equation (2.7) coincides on subdomain D_n of D with the solution given by formula (2.8).

1° For $n = 0$, we have

$$u(a, t) = u(a - t + t_0, t_0)e^{-\lambda(t-t_0)} \qquad (a, t) \in D_0,$$

since $u(a, t)$ satisfies Von Foerster equation (F). Initial condition (I) for u implies

$$\begin{aligned} u(a - t + t_0, t_0) &= \varphi(a - t + t_0) \\ &= \varphi(t_0 - b_0(t - a)). \end{aligned}$$

Hence, we obtain

$$u(a, t) = \varphi(t_0 - b_0(t - a))e^{-\lambda(t-t_0)}.$$

2° Suppose that for $(a, t) \in D_{n-1}$

(2.9) $u(a, t) = 2^{n-1}D_+b_{n-1}(t - a)\varphi(t_0 - b_{n-1}(t - a))e^{-\lambda(t-t_0)}$.

For $t_{n-1} \leqslant t < t_n$, we have $(a_g(t), t) \in D_{n-1}$. Then, by induction hypothesis (2.9),

$$u(a_g(t), t) = 2^{n-1}D_+b_{n-1}(t - a_g(t))\varphi(t_0 - b_{n-1}(t - a_g(t)))e^{-\lambda(t-t_0)}$$

$$= 2^{n-1}D_+b_{n-1}(b(t))\varphi(t_0 - b_{n-1}(b(t)))e^{-\lambda(t-t_0)} .$$

On the other hand, using boundary condition (B)

$$\begin{aligned} u(0, t) &= 2(1 - D_+a_g(t))u(a_g(t), t) \\ &= 2D_+b(t)\, 2^{n-1}D_+b_{n-1}(b(t))\varphi(t_0 - b_{n-1}(b(t)))e^{-\lambda(t-t_0)} \\ &= 2^nD_+b_n(t)\varphi(t_0 - b_n(t))e^{-\lambda(t-t_0)} . \end{aligned}$$

Therefore, for $(a, t) \in D_n$,

$$\begin{aligned} u(a, t) &= u(0, t - a)e^{-\lambda a} \\ &= 2^nD_+b_n(t - a)\varphi(t_0 - b_n(t - a))e^{-\lambda(t-a-t_0)}e^{-\lambda a} \\ &= 2^nD_+b_n(t - a)\varphi(t_0 - b_n(t - a))e^{-\lambda(t-t_0)} \end{aligned}$$

Q. E. D.

§ 3. Total Number of the Cells

It is very difficult, at the present level of experimental technique, to measure generation time $a_g(t)$ as a function of t. When one analyzes experimental data, it is, therfore, necessary to find $a_g(t)$ from another relation :

(2.10) $$N_V(t) = \int_0^{a_g(t)} u(a, t)da ,$$

where $N_V(t)$ represents the total number of the viable cells at time t, a quantity measurable by experiments. The following theorem gives a relation between $N_V(t)$ and $a_g(t)$ through the intermediary of $b_n(t)$.

Theorem 2.

In addition to the assumptions in Theorem 1, suppose that D_+a_g is right continuous and that φ is a non-negative, left continuous and integrable function on $(0, a_g(t_0)]$. Then, for any $t \geqslant t_0$, $u(a, t)$ given by formula (2.8) is an integrable function of a on $[0, a_g(t)]$ and its integral $N_V(t)$ defined by (2.10) satisfies the following formula :

(2.11) $$N_V(t) = \{2^n N_V(t_0) - 2^{n-1}\Phi(t_0 - b_n(t))\}e^{-\lambda(t-t_0)} \quad \text{for } t_{n-1} \leqslant t < t_n,$$

where $$\Phi(a) = \int_0^a \varphi(\alpha)\,d\alpha \qquad (0 \leqslant a \leqslant a_g(t_0)).$$

In order to prove Theorem 2, we prepare

Lemma .

Let F be a real-valued monotone increasing continuous function on $[a_0, a_1]$. Moreover, suppose that F is left differentiable on $(a_0, a_1]$ and that its left derivative D_-F is left continuous. Then,

$$\int_{a_0}^{a_1} D_-F(a)\,da = F(a_1) - F(a_0) \quad .$$

Proof of Lemma

Since F is a monotone increasing function, it is differentiable almost everywhere and satisfies

$$\int_{a_0}^{a_1} DF(a)\,da \leqslant F(a_1) - F(a_0) \quad ,$$

where $DF(a)$ denotes the derivative of F for almost all $a \in [a_0, a_1]$. (See e.g. [5] .) Since we have

$$0 \leqslant \int_{a_0}^{a_1} D_-F(a)\,da = \int_{a_0}^{a_1} DF(a)\,da \leqslant F(a_1) - F(a_0) < \infty ,$$

$f(a) = D_-F(a)$ is integrable on $(a_0, a_1]$.

$$G(x) = \int_{a_0}^{x} f(a)da$$

is, therefore, a continuous function of $x \in [a_0, a_1]$. Since f is left continuous by assumption, G is left differentiable and its left derivative D_-G is equal to f. Hence,

$$D_-(F - G) = 0 \quad \text{on } (a_0, a_1] .$$

F - G is, therefore, a constant on $[a_0, a_1]$, because F - G is continuous on $[a_0, a_1]$. Since $G(a_0) = 0$, this constant must be $F(a_0)$. So, we obtain

$$\int_{a_0}^{a_1} f(a)da = G(a_1) = F(a_1) - F(a_0)$$

Q. E. D.

Proof of Theorem 2

For $t_{n-1} \leqslant t < t_n$, we have

$$0 \leqslant t - t_{n-1} < a_g(t) .$$

Let us fix t for the moment and let F(a) denote $\Phi(t_0 - b_n(t - a))$ for $0 \leqslant a \leqslant t - t_{n-1}$. Since Φ and b_n are both continuous, F is also continuous. F is monotone increasing because $\varphi \geqslant 0$ and b_n is monotone increasing. As φ is left continuous, Φ is left differentiable and $D_-\Phi = \varphi$. Since b_n is strictly increasing and right differentiable, F is left differentiable and

$$D_-F(a) = \varphi(t_0 - b_n(t - a))D_+b_n(t - a)$$

which is left continuous, because of the left continuity of φ and the right continuity of D_+b_n. By Lemma, therefore,

$$\int_{0}^{t-t_{n-1}} D_-F(a)da = F(t - t_{n-1}) - F(0) ,$$

that is to say

$$\int_{0}^{t-t_{n-1}} \varphi(t_0 - b_n(t - a))D_+b_n(t - a)da$$

$$= \Phi(t_0 - b_n(t_{n-1})) - \Phi(t_0 - b_n(t)) .$$

Hence, we obtain

$$\int_0^{t-t_{n-1}} u(a, t)da = 2^n\{\Phi(a_g(t_0)) - \Phi(t_0 - b_n(t))\}e^{-\lambda(t-t_0)},$$

where we used relation (2.5). Similarly, we obtain

$$\begin{aligned}\int_{t-t_{n-1}}^{a_g(t)} u(a, t)da &= 2^{n-1}\{\Phi(t_0 - b_n(t)) - \Phi(t_0 - b_{n-1}(t_{n-1}))\}e^{-\lambda(t-t_0)} \\ &= 2^{n-1}\Phi(t_0 - b_n(t))e^{-\lambda(t-t_0)} \quad \text{(by (2.6))}.\end{aligned}$$

We have, therefore,

$$\begin{aligned}N_v(t) &= \int_0^{t-t_{n-1}} u(a, t)da + \int_{t-t_{n-1}}^{a_g(t)} u(a, t)da \\ &= \{2^n\Phi(a_g(t_0)) - 2^n\Phi(t_0 - b_n(t)) + 2^{n-1}\Phi(t_0 - b_n(t))\}e^{-\lambda(t-t_0)} \\ &= \{2^n\Phi(a_g(t_0)) - 2^{n-1}\Phi(t_0 - b_n(t))\}e^{-\lambda(t-t_0)} .\end{aligned}$$

When $t = t_0$, n is equal to 1 and we have

$$N_v(t_0) = 2\Phi(a_g(t_0)) - \Phi(t_0 - b(t_0)) = \Phi(a_g(t_0)) .$$

Hence,

$$N_v(t) = \{2^n N_v(t_0) - 2^{n-1}\Phi(t_0 - b_n(t))\}e^{-\lambda(t-t_0)} .$$

Q. E. D.

In the culture, dead cells accumulate gradually. Let $N_d(t)$ be the number of the dead cells at time t. In practice, counting the total number of the viable and dead cells i.e.

$$N(t) = N_v(t) + N_d(t)$$

is much easier than counting $N_v(t)$. It is clear that $N_d(t)$ is given by

$$\text{(2.12)} \qquad N_d(t) = N_d(t_0) + \lambda\int_{t_0}^{t} N_v(\tau)d\tau .$$

Hence, we have for $t_{n-1} \leqslant t < t_n$,

$$N_d(t) = N_d(t_0) + \lambda\{\sum_{k=1}^{n-1}\int_{t_{k-1}}^{t_k} N_v(\tau)d\tau + \int_{t_{n-1}}^{t} N_v(\tau)d\tau\}$$

$$= N_d(t_0) + \lambda \left[\sum_{k=1}^{n-1} \int_{t_{k-1}}^{t_k} \{2^k N_v(t_0) - 2^{k-1}\Phi(t_0 - b_k(\tau))\} e^{-\lambda(\tau - t_0)} d\tau \right.$$

$$\left. + \int_{t_{n-1}}^{t} \{2^n N_v(t_0) - 2^{n-1}\Phi(t_0 - b_n(\tau))\} e^{-\lambda(\tau - t_0)} d\tau \right]$$

by (2.11). We obtain finally

$$(2.13) \quad N(t) = N_d(t_0) + N_v(t_0)\{2 + \sum_{k=1}^{n-1} 2^k e^{-\lambda(t_k - t_0)}\}$$

$$- 2^{n-1}\Phi(t_0 - b_n(t)) e^{-\lambda(t - t_0)}$$

$$- \lambda\{ \sum_{k=1}^{n-1} 2^{k-1} \int_{t_{k-1}}^{t_k} \Phi(t_0 - b_k(\tau)) e^{-\lambda(\tau - t_0)} d\tau$$

$$+ 2^{n-1} \int_{t_{n-1}}^{t} \Phi(t_0 - b_n(\tau)) e^{-\lambda(\tau - t_0)} d\tau\} \qquad (t_{n-1} \leqslant t < t_n) .$$

§ 4. Biomass of the cells

Suppose that all the cells of age a have the same biomass $m(a, t)$ at time t. From the biological point of view as in § 1, $m(a, t)$ is considered a positive function on D which is differentiable in the direction of characteristic line of Von Foerster equation. Moreover, we suppose that all the cells increase their mass at a constant rate μ :

$$(2.14) \quad D_c m(a, t) = \mu m(a, t) .$$

Let $\psi(a)$ be the biomass of a cell of age a at the initial time t_0 :

$$(2.15) \quad m(a, t_0) = \psi(a) \qquad (0 < a \leqslant a_g(t_0)).$$

From assumption (3) in § 1, a cell having reached its generation time $a_g(t)$ at time t divides into two sister cells of equal biomass. Hence, we have a boundary condition for $m(a, t)$:

$$(2.16) \quad m(0, t) = \frac{1}{2} m(a_g(t), t) \qquad (t \geqslant t_0) .$$

Theorem 3

Let ψ be a real-valued function on $(0, a_g(t_0)]$. Then, there exists a unique solution $m(a, t)$ on D of the equation

$$(2.17)\ \begin{cases} (2.14) & D_c m(a, t) = \mu m(a, t) & \\ (2.15) & m(a, t_0) = \psi(a) & (0 < a \leqslant a_g(t_0)) \\ (2.16) & m(0, t) = \frac{1}{2} m(a_g(t), t) & (t \geqslant t_0) . \end{cases}$$

The solution $m(a, t)$ is given by the following formula :

$$(2.18)\quad m(a, t) = \frac{1}{2^n} \psi(t_0 - b_n(t - a))e^{\mu(t-t_0)} \qquad (a, t) \in D_n .$$

Proof

(Existence) It is shown similarly as in Theorem 1 that $m(a, t)$ given by formula (2.18) is a well-defined function on D and satisfies (2.14) and (2.15). As for boundary condition (2.16), we take a unique integer $n \geqslant 1$ such that $t_{n-1} \leqslant t < t_n$. From the relation $(0, t) \in D_n$, we have

$$m(0, t) = \frac{1}{2^n} \psi(t_0 - b_n(t))e^{\mu(t-t_0)} .$$

From the relation $(a_g(t), t) \in D_{n-1}$, we have

$$m(a_g(t), t) = \frac{1}{2^{n-1}} \psi(t_0 - b_{n-1}(b(t)))e^{\mu(t-t_0)}.$$

Hence, boundary condition (2.16) holds.

(Uniqueness) We shall show by mathematical induction on n that any solution $m(a, t)$ on D of equation (2.17) coincides on D_n with the solution given by (2.18).

1° For $n = 0$, we have, by (2.14),

$$m(a, t) = m(a - t + t_0, t_0)e^{\mu(t-t_0)} \qquad (a, t) \in D_0 .$$

By initial condition (2.15), we have

$$m(a - t + t_0, t_0) = \psi(a - t + t_0) .$$

Hence, we obtain

$$m(a, t) = \psi(t_0 - b_0(t - a))e^{\mu(t-t_0)} \qquad \text{for } (a, t) \in D_0 .$$

2° Suppose that $m(a, t)$ is given on D_{n-1} by (2.18). Using (2.16) and induction hypothesis, we have, for $t_{n-1} \leqslant t < t_n$,

$$m(0, t) = \frac{1}{2} m(a_g(t), t)$$
$$= \frac{1}{2} \frac{1}{2^{n-1}} \psi(t_0 - b_{n-1}(t - a_g(t)))e^{\mu(t-t_0)}$$
$$= \frac{1}{2^n} \psi(t_0 - b_n(t))e^{\mu(t-t_0)},$$

since $(a_g(t), t) \in D_{n-1}$. For any $(a, t) \in D_n$, the relation $t_{n-1} \leqslant t - a < t_n$ implies

$$m(a, t) = m(0, t - a)e^{\mu a}$$
$$= \frac{1}{2^n} \psi(t_0 - b_n(t - a))e^{\mu(t - a - t_0)}e^{\mu a}$$
$$= \frac{1}{2^n} \psi(t_0 - b_n(t - a))e^{\mu(t-t_0)}.$$

Q. E. D.

If D_+a_g is right continuous and $\psi(a) \varphi(a)$ is a non-negative, left continuous and integrable function on $(0, a_g(t_0)]$, then, for $t \geqslant t_0$, $m(a, t)u(a, t)$ is an integrable function of a on $[0, a_g(t)]$ and its integral

$$M_V(t) = \int_0^{a_g(t)} m(a, t)u(a, t)da$$

represents the total biomass of the viable cells at time t. We shall calculate $M_V(t)$, using the results (2.8) in Theorem 1 and (2.18) in Theorem 3. For $t_{n-1} \leqslant t < t_n$, we have

$$\int_0^{t-t_{n-1}} m(a, t)u(a, t)da$$

$$= \int_0^{t-t_{n-1}} D_+b_n(t - a)\psi(t_0 - b_n(t - a)) \varphi(t_0 - b_n(t - a))e^{(\mu-\lambda)(t-t_0)}da$$

$$= \Big[\Psi(t_0 - b_n(t - a))\Big]_{a=0}^{a=t-t_{n-1}} e^{(\mu-\lambda)(t-t_0)} ,$$

where $\Psi(a) = \int_0^a \psi(\alpha) \varphi(\alpha)d\alpha$. Similarly we have

$$\int_{t-t_{n-1}}^{a_g(t)} m(a, t)u(a, t)da$$

$$= \left[\Psi(t_0 - b_{n-1}(t - a))\right]_{a=t-t_{n-1}}^{a=a_g(t)} e^{(\mu-\lambda)(t-t_0)} .$$

Hence, we obtain

$$M_v(t) = \Psi(a_g(t_0))e^{(\mu-\lambda)(t-t_0)} = M_v(t_0)e^{(\mu-\lambda)(t-t_0)} , \tag{2.19}$$

which is the formula to be expected naturally. (2.19) is consistent with experimental data, which justifies the whole framework of our theory.

Let $M_d(t)$ be the total biomass of the dead cells at time t. Then we have

$$\begin{aligned} M_d(t) &= M_d(t_0) + \lambda\int_{t_0}^{t} M_v(\tau)d\tau \\ &= \frac{\lambda}{\mu - \lambda} M_v(t_0)e^{(\mu-\lambda)(t-t_0)} + M_d(t_0) - \frac{\lambda}{\mu - \lambda} M_v(t_0) . \end{aligned} \tag{2.20}$$

Hence, the total biomass $M(t)$ of the viable and dead cells at time t is given by

$$\begin{aligned} M(t) &= M_v(t) + M_d(t) \\ &= \frac{\mu}{\mu - \lambda} M_v(t_0)e^{(\mu-\lambda)(t-t_0)} + M_d(t_0) - \frac{\lambda}{\mu - \lambda} M_v(t_0) . \end{aligned} \tag{2.21}$$

Chapter 3. Some Special Cases and Applications

§ 1. The Case in Which $a_g(t)$ Is Constant

In this section we suppose that generation time $a_g(t)$ is independent of time t and we denote its constant value by a_g. It is evident that $a_g(t) \equiv a_g$ satisfies all the assumptions on $a_g(t)$ in Theorems 1 and 2. By definition of $b(t)$, we have

$$b(t) = t - a_g .$$

Hence, we obtain

$$t_n = t_0 + na_g , \qquad b_n(t) = t - na_g .$$

Boundary condition (B) reads, in this case, as follows :

$$u(0, t) = 2u(a_g, t)$$

(see Scherbaum and Rasch[6]). Thus the solution $u(a, t)$ of (2.7) is written as

(3.1) $\quad u(a, t) = 2^n \varphi(a - t + t_n)e^{-\lambda(t-t_0)} \quad$ for $(a, t) \in D_n$.

The total number $N_V(t)$ of the viable cells at time t is given by

(3.2) $\quad N_V(t) = \{2^n N_V(t_0) - 2^{n-1}\Phi(t_n - t)\}e^{-\lambda(t-t_0)} \quad$ for $t_{n-1} \leqslant t < t_n$,

where $\Phi(a)$ is the integral of φ on $(0, a]$, φ being any non-negative left continuous integrable function defined on $(0, a_g]$. Even if φ is continuous on the whole interval $(0, a_g]$, the solution $u(a, t)$ is not necessarily continuous on characteristic lines $t - a = t_{n-1}$ $(n \geqslant 1)$. In this connection we have the following proposition.

<u>Proposition 1</u>

The following conditions (i) and (ii) are equivalent.

(i) $u(a, t)$ given by (3.1) is right continuous with respect to a at (a, t) such that $t - a = t_{n-1}$.

(ii) $\lim_{h \downarrow 0} \varphi(h) = 2\varphi(a_g)$.

<u>Proof</u>

When $t - a = t_{n-1}$, $(a + h, t)$ belongs to D_{n-1} for $0 < h \leqslant a_g - a$. We have, therefore,

$$\begin{aligned} u(a + h, t) &= 2^{n-1}\varphi(a + h - t + t_{n-1})e^{-\lambda(t-t_0)} \\ &= 2^{n-1}\varphi(h)e^{-\lambda(t-t_0)} . \end{aligned}$$

Hence, we have

$$u(a + h, t) - u(a, t) = 2^{n-1}\{\varphi(h) - 2\varphi(a_g)\}e^{-\lambda(t-t_0)} ,$$

which leads to the desired equivalence.

Q. E. D.

From Proposition 1 and the continuity of u in the direction of characteristic line, we have the following proposition.

Proposition 2

The following conditions (i) and (ii) are equivalent.

(i) u given by (3.1) is continuous on D.

(ii) φ is continuous on $(0, a_g]$ and $\lim_{h \downarrow 0} \varphi(h) = 2\varphi(a_g)$.

§ 2. Logarithmic Phase (Period of Steady Growth)

When there is no limiting factor in batch culture, the cells enter sooner or later into a period in which they grow steadily at their own maximum growth rate. The period with this aspect of growth is called conventionally "logarithmic phase" by microbiologists.

In logarithmic phase the age distribution is considered to be stable in shape, which means

(i) $a_g(t)$ is constant $(= a_g)$,

and

(ii) $u(a, t)$ can be written as

$$u(a, t) = A(a)T(t) ,$$

where A is a non-negative function defined on $[0, a_g]$ and T is a positive one on $[t_0, \infty)$.

Since $u(a, t)$ is the density function of cell number with respect to a, $A(a)$ must be integrable on $[0, a_g]$, and the total number $N_V(t)$ of the viable cells is given by

$$N_V(t) = T(t)\int_0^{a_g} A(a)da$$

$T(t)$ is, therefore, considered to be no less differentiable than $N_V(t)$. From the relation

$$\{u(a + h, t + h) - u(a, t)\}/h$$
$$= T(t + h)\{A(a + h) - A(a)\}/h + A(a)\{T(t + h) - T(t)\}/h ,$$

$A(a)$ is differentiable if $u(a, t)$ is differentiable in the direction of characteristic line. Hence, when u satisfies Von Foerster equation (F), we have

$$\frac{d}{da} A(a)\ T(t) + A(a) \frac{d}{dt} T(t) = -\lambda A(a)T(t) \quad .$$

Thus we find

$$-\frac{1}{A(a)} \frac{d}{da} A(a) = \frac{1}{T(t)} \frac{d}{dt} T(t) + \lambda \quad .$$

Both sides of this equation must equal a positive constant μ', and so

(3.3) $\qquad u(a, t) = u_0 e^{\mu'(t-a)-\lambda(t-t_0)} \quad ,$

where u_0 is a positive constant. This is a solution of Von Foerster equation without initial condition or boundary condition. If we take $t = t_0$,

(3.4) $\qquad u(a, t_0) = u_0 e^{\mu'(t_0-a)} \quad .$

Conversely, suppose that $u(a, t)$ is a solution of Von Foerster equation (F) with initial condition (3.4) and boundary condition (B) with constant a_g. Then we can apply Theorems 1 and 2 to this case taking $\varphi(a) = u_0 e^{\mu'(t_0-a)}$ $(0 < a \leq a_g)$ and obtain from (3.1) and (3.2) in §1

(3.5) $\qquad u(a, t) = (2e^{-\mu' a_g})^n u_0 e^{\mu'(t-a)-\lambda(t-t_0)} \qquad$ for $(a, t) \in D_n$

(3.6) $\qquad N_V(t) = \dfrac{u_0 2^{n-1}}{\mu'} \{(e^{-\mu' a_g})^n e^{\mu' t} + (1 - 2e^{-\mu' a_g}) e^{\mu' t_0}\} e^{-\lambda(t-t_0)}$

for $t_{n-1} \leq t < t_n$.

The necessary and sufficient condition for continuity of u given by (3.5) is by Proposition 2

$$\lim_{h \downarrow 0} \varphi(h) = 2\varphi(a_g) \quad .$$

This condition is equivalent to

(3.7) $\qquad \mu' = \dfrac{\log 2}{a_g} \quad .$

Substituting (3.7) into (3.5) and (3.6), we obtain

(3.8) $\qquad u(a, t) = u_0 e^{\mu'(t-a)-\lambda(t-t_0)}$

(3.9) $\qquad N_V(t) = \dfrac{u_0}{2\mu'} e^{\mu' t-\lambda(t-t_0)}$

So we have by (2.12)

$$N_d(t) = N_d(t_0) + \frac{\lambda u_0}{2\mu'(\mu'-\lambda)}(e^{\mu' t-\lambda(t-t_0)} - e^{\mu' t_0}) \quad .$$

Hence

$$(3.10) \quad N(t) = N_v(t) + N_d(t)$$
$$= \frac{u_0}{2(\mu'-\lambda)} e^{\mu' t-\lambda(t-t_0)} + N_d(t_0) - \frac{\lambda u_0}{2\mu'(\mu'-\lambda)} e^{\mu' t_0}$$
$$= \frac{\mu'}{\mu'-\lambda} N_v(t_0) e^{(\mu'-\lambda)(t-t_0)} + N_d(t_0) - \frac{\lambda}{\mu'-\lambda} N_v(t_0) \quad .$$

As for the biomass in logarithmic phase, $m(a, t)$ is considered independent of t. Partial differential equation (2.14) reduces, then, to ordinary differential equation

$$\frac{dm}{da} = \mu m$$

and we obtain

$$(3.11) \quad m(a, t) \equiv m(a) = m_0 e^{\mu a} \quad ,$$

with a positive constant m_0 . Boundary condition (2.16) for $m(a, t)$ implies

$$(3.12) \quad \mu = \frac{\log 2}{a_g} \quad ,$$

which means

$$\mu = \mu' \quad .$$

§ 3. Transition Phase

We take the end point of logarithmic phase for the initial time t_0 and suppose that $a_g(t)$ depends on t. In other words we consider equation (2.7) with $\varphi(a) = u_0 e^{\mu(t_0-a)}$. Then we have by Theorems 1 and 2

$$(3.13) \quad u(a, t) = 2^n u_0 D_+ b_n(t - a) e^{\mu b_n(t-a)-\lambda(t-t_0)} \qquad \text{for } (a, t) \in D_n \ ,$$

$$(3.14) \quad N_v(t) = \{2^n N(t_0) + 2^{n-1} \frac{u_0}{\mu} (e^{\mu b_n(t)} - e^{\mu t_0})\} e^{-\lambda(t-t_0)}$$
$$= \frac{2^{n-1} u_0}{\mu} e^{\mu b_n(t)-\lambda(t-t_0)} \qquad \text{for } t_{n-1} \leqslant t < t_n \ .$$

As for $m(a, t)$, taking $\psi(a) = m_0 e^{\mu a}$, we have by Theorem 3

$$(3.15) \qquad m(a,\ t) = \frac{m_0}{2^n} e^{\mu\{t-b_n(t-a)\}} \qquad \text{for } (a,\ t) \in D_n \quad .$$

Especially for $t_0 \leqslant t < t_1$ we have

$$(3.16) \qquad u(a,\ t) = \begin{cases} 2u_0\{1 - D_+a_g(t-a)\}e^{\mu\{t-a-a_g(t-a)\}-\lambda(t-t_0)} & 0 \leqslant a \leqslant t - t_0 \\ u_0 e^{\mu(t-a)-\lambda(t-t_0)} & t - t_0 < a \leqslant a_g(t) \end{cases}$$

$$(3.17) \qquad N_v(t) = \frac{u_0}{\mu} e^{\mu\{t-a_g(t)\}-\lambda(t-t_0)} \quad .$$

§ 4. Application to Experimental Data

The graphs of $\log M(t)$ and $\log N(t)$ against t in logarithmic phase are linear within the range of experimental errors, which is the origin of the name "logarithmic phase". This fact means that the constant terms of equations (2.21) and (3.10) are negligible compared to the time dependent exponential terms. According to (2.21) and (3.10), one can determine $\mu - \lambda$ as the common gradient of the graphs of $\log M(t)$ and $\log N(t)$, obtained through experiments. Observation of biomass $m(a)$ of individual cell as a function of cell age a may be possible in principle, which gives the value of μ by (3.11). Hence λ also can be determined.

In the transition phase, one can calculate $a_g(t)$ by (3.17) because $N_v(t)$ can be obtained through the experiment called "viable count". Thus $u(a,\ t)$ is calculable by (3.16).

We now apply the above theory to the experimental data obtained by Maruyama et al. [1]. Since the data concerning $m(a)$ is not available, we are obliged to guess the value of the death rate λ, which is considered to be not so significant in logarithmic phase, the period in which the cells grow in their best condition. In the transition phase which follows logarithmic phase, λ is still considered to remain small as the transition phase we treat is rather short i.e. less than about $1.5a_g(t_0)$. Hence we step forward, putting $\lambda = 0$.

Generation time $a_g(t)$ calculated by (3.17) from the data of Maruyama et al. shows that the culture reaches the starvation point t_s before the curve $a = a_g(t)$ crosses the characteristic line $t - a = t_0$. (3.16) and (3.17) are sufficient for the analysis of the transition phase in this case. Figure 2 shows the regression line for $a_g(t)$ thus obtained which is almost linear. Since the growth rate of cell number is given by

$$\frac{1}{N_V(t)} D_+N_V(t) = \mu(1 - D_+a_g(t)) \ ,$$

$D_+a_g(t) > 0$ implies that the growth rate in the transition phase is less than μ, that in logarithmic phase.

Since concentration C of the limiting nutrient (glucose) is given in the data of Maruyama et al. as a function of t, we can transform $a_g(t)$ into $a_g(C)$ (see Assumption (1)). The result shows that the critical concentration below which generation time depends on C is about 28 times as high as the value reported by Monod[7] with respect to *Escherichia coli*, which means that the cells in the case of Maruyama et al. are more susceptible to deficiency of nutrient than the cells in Monod's case. When C lowers below the critical concentration, $1/a_g$ decreases linearly against C.

Substituting the values of $a_g(t)$ in Figure 2 into equation (3.16), we can calculate $u(a, t)$ in the transition phase. The numerical calculation shows that a peak is formed in $u(a, t)$ with respect to a and the peak moves rightward (Figure 3). When the peak reaches $a_g(t)$, relatively large number of the cells divide simultaneously, i.e. the cell cycle of the cells are partially synchronized. In the case of Maruyama et al., as seen in Figure 3, the culture enters into stationary phase (which we do not treat in this note) before the peak reaches $a_g(t)$. Accordingly the synchronous division does not

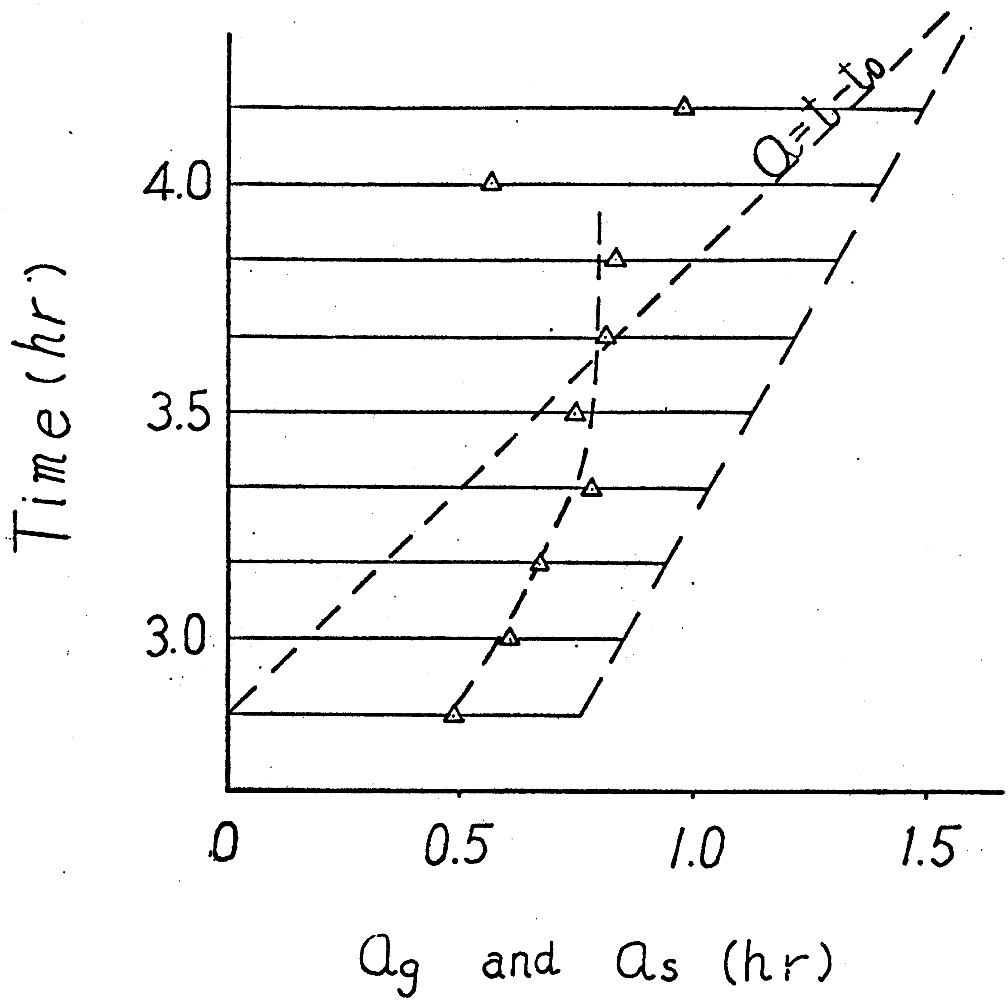

Figure 2. Variation of $a_s(t)$, as well as $a_g(t)$, with time. The right end of each horizontal line and the triangle (▲) on it represent $a_g(t)$ and $a_s(t)$ at the same t, respectively.

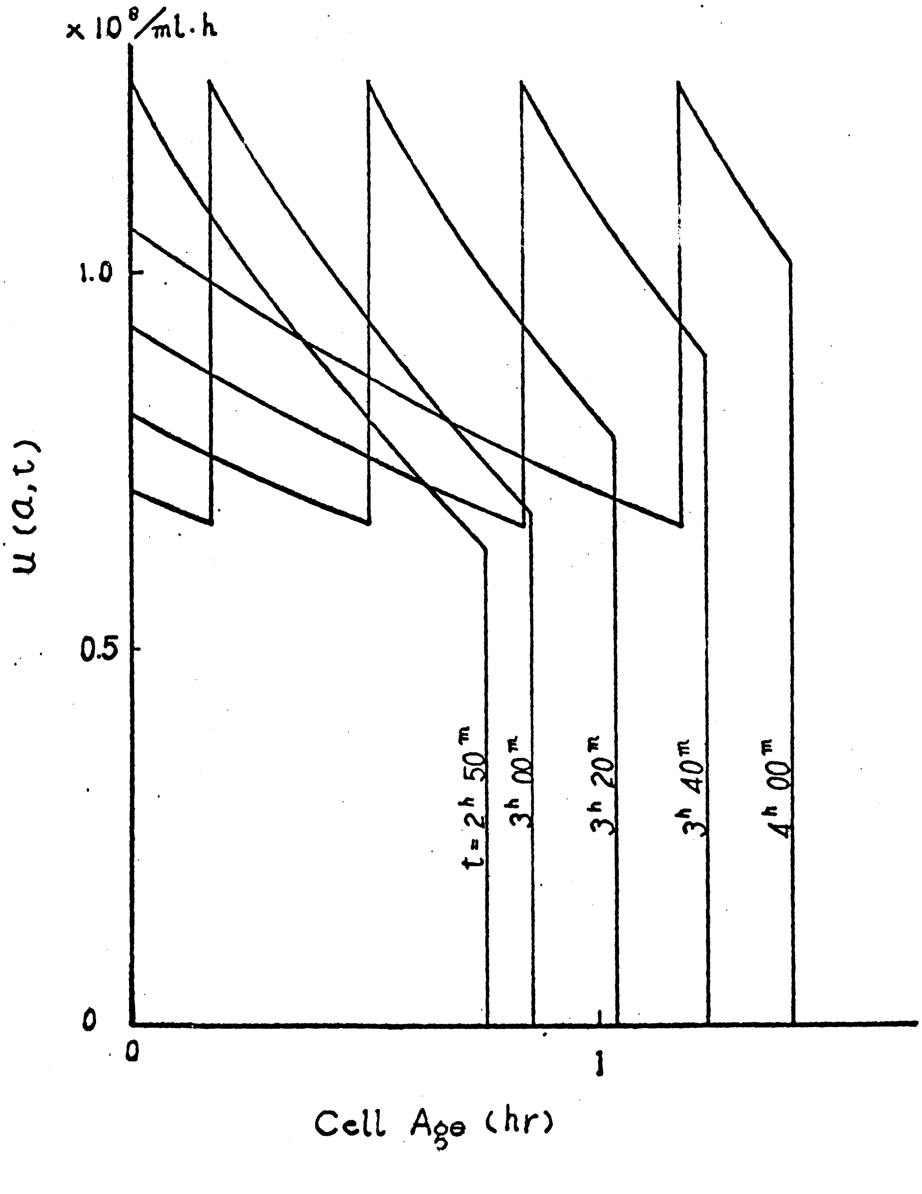

Figure 3. Movement of age distribution of the cells with time.

occur in the transition phase in their case. It is, however, clear from our calculation that the cells are already partially synchronized in this period. The formation of the peak in $u(a, t)$ is also, mathematically, due to the fact that $D_+a_g(t) > 0$.

When rod-like bacteria such as *Bacillus subtilis* grow, they generally increase their length, not their thickness. A septum appears, as they grow, at the middle of the length of a cell (then called "septated cell") and after some time division occurs at the site of the septum. As Maruyama et al. have measured the ratio of the number of the septated cells to the total number of the cells, the value of

$$\frac{\int_{a_s(t)}^{a_g(t)} u(a, t)da}{\int_{0}^{a_g(t)} u(a, t)da}$$

is known at each t, where $a_s(t)$ is the age of the youngest cells that have observable septum at time t. Hence, we can calculate $a_s(t)$. $a_s(t)$ is considered to have close relation to the cell age at which the biosynthesis of the septum starts. The result is shown in Figure 2. In the early transition phase $a_g(t) - a_s(t) \fallingdotseq$ const., which suggests that intracellular changes resulting in the elongation of $a_g(t)$ occur during $0 < a < a_s(t)$. In the late transition phase, on the other hand,

$$D_+a_g(t) > D_+a_s(t) \fallingdotseq 0 ,$$

which suggests that the intracellular changes occur principally in the cells of age $a > a_s(t)$.

Acknowledgement

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Note on the error estimate for the Newmark-β method applied to the second order linear evolution equation of hyperbolic type.

By Teruo USHIJIMA

Introduction.

In this note we present an error estimate for the Newmark-β method applied to the following initial value problem (E) of the 2nd order evolution equation of hyperbolic type in a Hilbert space X with the positive definite selfadjoint operator A as its coefficient.

$$(E)\quad \begin{cases} \dfrac{d^2u(t)}{dt^2} + Au(t) = 0, \qquad t>0, \\ u(0) = a^1, \quad (\dfrac{d}{dt}u)(0) = a^0. \end{cases}$$

Under two conditions (A) and (I), being specified in §1, it will be shown that the error measured in energy norm is O(h) if the stability condition is satisfied. Here condition (A) corresponds to the error estimate for the finite element solution of the stationary problem. And condition (I) is a kind of inverse assumption in the finite element approximation. The parameter h means the representative leghth of the triangulation.

The aim of this note is an operator theoretical reconstruction of the analysis done in Fujii [1] for the dynamic problem in the linear elasticity theory. Our method is based upon the approximation theory for semi-groups of linear operators due to Trotter-Kato (see e.g. Kato [3], see also Ushijima [5]). Hence our estimate is valid so far as the initial data a^1, and a^0, belonging to the domain $D(A^{3/2})$, and $D(A)$, respectively.

Our problem and main result is stated in §1. Then we establish an abstract theorem concerning the error estimate for approximate discrete semi-group in §2, using the results developed in Kato [3] and Ushijima [5]. The stability criterion of our scheme is discussed in §3. Finally the proof of main result is shown in §4 after series of Propositions are prepared.

In the anthor's previous work [6], we discussed the semi-discrete approximation of the problem (E). Namely the adopted approximate problem was also the Cauchy problem for the 2nd order ordinary differential equation. This note supplies a result concerning time discretization problem of (E). Due to the discussion given in §5 of [5], our result can be applied to the mixed problem with Dirichlet boundary condition for both the scalor wave equation and the system of linear elasticity.

The author would like to express his sincere thanks to his respected friend, Professor H.Fujii of Kyoto Sangyo University. Without his constant interest to this study with warm encouragement, this note could not be written.

§1. Setting of the problem and the main result.

Consider the problem (E) stated in Introduction of this note. The inner product, and the norm of X, is denoted by (,), and $\|\ \|$, respectively. Let V be the domain of the positive square root $A^{1/2}$ of A. The set V is considered as a Hilbert space with the inner product $(u,v)_V = (A^{1/2}u, A^{1/2}v)$ for $u,v \in V$.

Let X_h be a closed subspace of X contained in V, being dependent of positive parameter h. And let A_h be the Galerkin approximation of A. Namely A_h is the bounded selfadjoint operator acting in X_h defined by the formula :

$$(A_h u_h, v_h) = (u_h, v_h)_V \quad \text{for } u_h, v_h \in X_h.$$

Following to Newmark [4], we construct an X_h-valued approximate solution

$u_h(t)$ of (E) by the following scheme $(E^{\tau}_{h,\beta})$.

$$(E^{\tau}_{h,\beta}) \begin{cases} D_{\tau\bar{\tau}}u_h(t) + A_h u_h(t) + \beta\tau^2 D_{\tau\bar{\tau}}u_h(t) = 0, \\ \qquad\qquad n\tau \leqslant t < (n+1)\tau, \quad n=1,2,\cdots, \\ u_h(t) = a_h^1 , \qquad 0 \leqslant t < \tau, \\ \qquad = a_h^1 - \tau a_h^0 , \qquad -\tau \leqslant t < 0. \end{cases}$$

Here β is a nonnegative number, τ is positive, and $D_{\tau\bar{\tau}} = D_\tau D_{\bar{\tau}} = D_{\bar{\tau}} D_\tau$, where $(D_\tau u)(t) = \tau^{-1}(u(t+\tau) - u(t))$, $(D_{\bar{\tau}}u)(t) = (D_\tau u)(t-\tau)$.

Let P_{1h} , and P_{0h} , be the orthogonal projection onto X_h from the Hilbert space V, and X, respectively. We require the following two conditions.

(A) $\quad \| A^{1/2}(P_{1h}v - v)\| \leqslant Ch\| Av\|$, $\quad v \in D(A)$.

(I) $\quad h\| A^{1/2}v_h\| \leqslant C\| v_h\|$, $\quad v_h \in X_h$.

In the above two conditions, C denote the constants independent of h and, v or v_h.

<u>Theorem 1.</u> Assume (A) and (I). If $0 \leqslant \beta < 1/4$, fixing γ such that $0 < \gamma < \sqrt{4/(1-4\beta)}$, and if $\beta = 1/4$ fixing γ such that $0 < \gamma < \infty$, choose τ in $(E^{\tau}_{h,\beta})$ so as to satisfy $\tau\| A_h\|^{1/2} \leqslant \gamma$. If $\beta > 1/4$, choose τ being bounded in h. Let u(t) be the solution of (E) with the initial datum $a^1 \in D(A^{3/2})$ and $a^0 \in D(A)$, and let $u_h(t)$ be the solution of $(E^{\tau}_{h,\beta})$ with the initial datum $a_h^1 = P_{1h}a^1$ and $a_h^0 = P_{0h}a^0$. Then there is a constant C satisfying that

$$\| A^{1/2}(u_h(t) - u(t))\|^2 + \| D_{\bar{\tau}}u_h(t) - \frac{d}{dt}u(t)\|^2 \leqslant C(h^2 + \tau^2)\{\| A^{3/2}a^1\|^2 + \| Aa^0\|^2\} , \qquad 0 \leqslant t \leqslant T,$$

where C depends on $\beta,\gamma,T \sup_h \tau$, but not on a^1 and a^0.

In the above theorem,

$$\| A_h\| = \sup_{v_h \in X_h} \frac{\| A_h v_h\|}{\| v_h\|} ,$$

which is $O(h^{-2})$ as h tends to 0 by virtue of (I).

§2. Error estimate for approximate discrete semi-group.

First we briefly resume terminologies used in Kato [3] and Ushijima [5]. Let $T(t)$, $t\geqslant 0$, be a discrete semi-group with time unit τ acting in a Banach space x. Namely there is a bounded operator $T(\tau)$ acting in X with the property that $T(t) = (T(\tau))^{[t/\tau]}$ for $t\geqslant 0$. The operator $\tau^{-1}(T(\tau) - 1)$ is said to be the generator of discrete semi-group $T(t)$.

Consider a family of Banach spaces $\{X_h: h>0\}$ and a Banach space X. Let $T(t) = e^{-tA}$ be a continuous semi-group in X (of class C^0 in the usual terminology of semi-group theory, cf Hille-Phillips [2]). Suppose there is a discrete semi-group $T_h(t)$ acting in X_h with time unit τ_h for any h. Let A_h be the generator of $T_h(t)$. Assume that there exist continuous linear operators P_h from X into X_h, and a scalor function $\varepsilon(h,\tau)$ from $(0,\infty)\times(0,\infty)$ into $(0,\infty)$ so as to satisfy the following two conditions $(A_{\varepsilon,\tau})$ and (B_τ).

$(A_{\varepsilon,\tau})$ There exist bounded inverse $A_h^{-1} \in L(X_h)$, $h>0$, and $A^{-1} \in L(X)$ such that

$$\sup_{h>0} \| A_h^{-1} \|_{L(X_h)} < \infty$$

and that

$$\| A_h^{-1} P_h a - P_h A^{-1} a\|_{X_h} \leqslant \varepsilon(h,\tau_h) \| a\|_X , \quad a\in X .$$

(B_τ) There is a positive constant $\bar{\tau}$ such that

$$\tau_h \leqslant \bar{\tau} \quad \text{for } h>0,$$

and that

$$\sup_{0\leqslant t\leqslant \bar{\tau},\, h>0} \| T_h(t)\|_{L(X_h)} < \infty .$$

Theorem 2. There is a constant C such that

$$(2.1) \quad \| (T_h(t)P_h - P_h T(t))a\|_{X_h} \leqslant C(\varepsilon(h,\tau_h) + \tau_h \| P_h\|_{L(X,X_h)}) \| A^2 a\|_X, \quad 0\leqslant t\leqslant T,\ a\in D(A^2).$$

Proof. Condition (B_τ) implies that there are positive constants N and ω such that

$$\| T_h(t) \|_{L(X_h)} \leqslant Ne^{\omega t} \qquad \text{for } t\geqq 0 \text{ and } h>0.$$

Due to the estimation method developed in Chapt. IX, §3.1 of Kato [3], we have

$$(2.2) \qquad \| T_h(t) - e^{-tA_h} \|_{L(X_h)} \leqslant \tau_h e^{\bar{\omega}t} \{ \frac{N^2}{2} t \| A_h{}^2 \|_{L(X_h)} + N \| A_h \|_{L(X_h)} \} \qquad \text{for } t\geqq 0,$$

with

$$\bar{\omega} = \sup_{h>0} \tau_h{}^{-1} (e^{\tau_h \omega} - 1) ,$$

and

$$(2.3) \qquad \| e^{-tA_h} \|_{L(X_h)} \leqslant Ne^{\bar{\omega}t} \qquad \text{for } t\geqq 0.$$

From Theorem 1 of Ushijima [6], $(A_{\varepsilon,\tau})$ and (2.3) assure the following estimate (2.4).

$$(2.4) \qquad \| (e^{-tA_h}P_h - P_h T(t))a \|_{X_h} \leqslant C_T \varepsilon(h,\tau_h) \| A^2 a \|_X , \qquad 0\leqslant t\leqslant T,\ a\in D(A^2),$$

where C_T means a constant dependent of T but independent of h and a.

We have

$$\| (T_h(t) - e^{-tA_h}) P_h A^{-2} \|_{L(X,X_h)}$$

$$\leqslant \| (T_h(t) - e^{-tA_h}) A_h{}^{-2} P_h \|_{L(X,X_h)}$$

$$+ \| (T_h(t) - e^{-tA_h}) (A_h{}^{-2} P_h - P_h A^{-2}) \|_{L(X,X_h)} .$$

In the right hand side of this inequality, the first term is majorized by $C_T \tau_h \| P_h \|_{L(X,X_h)}$ from (2.2), and the second term by $C_T \varepsilon(h,\tau_h)$ from $(A_{\varepsilon,\tau})$.

So we have

$$(2.5) \qquad \| (T_h(t) - e^{-tA_h}) P_h A^{-2} \|_{L(X,X_h)}$$

$$\leqslant C \{ \tau_h \| P_h \|_{L(X,X_h)} + \varepsilon(h,\tau_h) \} , \qquad 0\leqslant t\leqslant T.$$

Finally it holds good that

$$\| (T_h(t) P_h - P_h T(t)) a \|_{X_h}$$

$$\leqslant \| (T_h(t) - e^{-tA_h}) P_h A^{-2} A^2 a \|_{X_h} + \| (e^{-tA_h} P_h - P_h T(t)) a \|_{X_h}$$

$$\leqslant (\text{ by (2.5) and (2.4) })$$

$$\leqslant C\,(\tau_h \| P_h \|_{L(X,X_h)} + \varepsilon(h,\tau_h)) \| A^2 a \|_X .$$

Hence we have (2.1).

§3. Stability of the scheme.

Let us introduce the bounded positive selfadjoint operator $A_{h,\beta}$ defined by

(3.1) $\quad A_{h,\beta} = (1 + \beta\tau^2 A_h)^{-1} A_h .$

Then our scheme $(E^{\tau}_{h,\beta})$ can be rewritten in the following form (3.2).

$$(3.2) \quad \begin{cases} D_{\tau\bar{\tau}} u_h(t) + A_{h,\beta} u_h(t) = 0, \\ \qquad\qquad n\tau \leqslant t < (n+1)\tau,\ n=0,1,2,\cdots, \\ u_h(t) = a_h^1 , \quad 0 \leqslant t < \tau, \\ \qquad\ = a_h^1 - \tau a_h^0 , \quad -\tau \leqslant t < 0. \end{cases}$$

This scheme (3.2) can be transformed to the following one step scheme $(\mathbb{E}^{\tau}_{h,\beta})$.

$$(\mathbb{E}^{\tau}_{h,\beta}) \quad \begin{cases} \mathbf{u}_h(t+\tau) = (1 - \tau\mathbb{A}_{h,\beta} - \tau^2 \mathbb{B}_{h,\beta}) \mathbf{u}_h(t) , \\ \qquad\qquad n\tau \leqslant t < (n+1)\tau,\ n=1,2,\cdots, \\ \mathbf{u}_h(t) = \mathbf{a}_h = \mathbb{P}_h \mathbf{a}, \quad 0 \leqslant t < \tau . \end{cases}$$

Here $\mathbf{u}_h = \binom{u_h(t)}{D_{\bar{\tau}} u_h(t)}$ is considered to be an element of the product space $\mathbf{X}_h = \begin{smallmatrix} X_h \\ \times \\ X_h \end{smallmatrix}$, $\mathbb{P}_h$ is the operator from the product Hilbert space $\mathbb{X} = \begin{smallmatrix} V \\ \times \\ X \end{smallmatrix}$ onto $\mathbf{X}_h$ defined by the following matrix expression:

$$\mathbb{P} = \begin{pmatrix} P_{1h} & 0 \\ 0 & P_{0h} \end{pmatrix} ,$$

$\mathbb{A}_{h,\beta}$ and $\mathbb{B}_{h,\beta}$ are bounded linear operators acting on $\mathbf{X}_h$ defined by

$$\mathbb{A}_{h,\beta} = \begin{pmatrix} 0 & -1 \\ A_{h,\beta} & 0 \end{pmatrix} , \quad \mathbb{B}_{h,\beta} = \begin{pmatrix} A_{h,\beta} & 0 \\ 0 & 0 \end{pmatrix} .$$

Let $V_{h,\beta}$ be the Hilbert space X_h with the inner product:

$$(a_h, b_h)_{h,\beta} = (A_{h,\beta}^{1/2} a_h, A_{h,\beta}^{1/2} b_h) \quad \text{for} \quad a_h, b_h \in X_h,$$

and let $\mathbf{X}_{h,\beta} = \begin{smallmatrix} V_{h,\beta} \\ \times \\ X_h \end{smallmatrix}$ be the product Hilbert space, whose norm is determined by

$$\| \mathbf{a}_h \|_{h,\beta} = (\| A_{h,\beta}{}^{1/2} a^1{}_h \|^2 + \| a^0{}_h \|^2)^{1/2}$$
$$\text{for } \mathbf{a}_h = \begin{pmatrix} a_h^1 \\ a_h^0 \end{pmatrix} \in \mathbf{X}_{h,\beta} \quad .$$

Now we consider discrete semi-groups $\mathbf{T}_{h,\beta}(t)$ acting on $\mathbf{X}_{h,\beta}$ defined by

$$\mathbf{T}_{h,\beta}(t) = [1 - \tau(\mathbb{A}_{h,\beta} + \mathbb{B}_{h,\beta})]^{[t/\tau]} , \qquad t \geqslant 0.$$

Proposition 3. If $0 \leqslant \beta < 1/4$, fix γ satisfying

$$(3.3) \quad 0 < \gamma < \sqrt{\frac{4}{1 - 4\beta}} \quad ,$$

and if $\beta = 1/4$, fix γ arbitrary positive constant. Define $\tilde{\gamma}$ by

$$(3.4) \quad \tilde{\gamma} = \begin{cases} \dfrac{\gamma}{\sqrt{1 + \beta\gamma^2}} & \text{if } 0 \leqslant \beta \leqslant 1/4 , \\ \dfrac{1}{\sqrt{\beta}} & \text{if } \quad \beta > 1/4 . \end{cases}$$

Choose τ in $(\mathbb{E}_{h,\beta})$ so as to satisfy

$$(3.5) \quad \tau(\| A_h \|_{L(X_h)})^{1/2} \leqslant \gamma \qquad \text{if } 0 \leqslant \beta \leqslant 1/4 ,$$

being arbitrary if $\beta > 1/4$. Then we have the estimate

$$(3.6) \quad \| \mathbf{T}_{h,\beta}(t) \mathbf{a}_h \|_{h,\beta} \leqslant \sqrt{\frac{1 + \tilde{\gamma}/2}{1 - \tilde{\gamma}/2}} \, \| \mathbf{a}_h \|_{h,\beta} , \qquad t \geqslant 0 .$$

Proof. Since A_h is bounded positive definite selfadjoint, we have

$$\| A_{h,\beta} \|_{L(X_h)} = (1 + \beta\tau^2\alpha_h)^{-1}\alpha_h$$

with $\alpha_h = \| A_h \|_{L(X_h)}$. Hence

$$\begin{aligned} &\tau(\| A_{h,\beta} \|_{L(X_h)})^{1/2} \\ &= (1 + \beta(\tau\sqrt{\alpha_h})^2)^{-1/2} \cdot \tau\sqrt{\alpha_h} \\ &\leqslant (\text{ by } (3.5)) \\ &\leqslant \gamma/\sqrt{1 + \beta\gamma^2} \\ &\leqslant (\text{ by } (3.4)) \\ &\leqslant \tilde{\gamma} . \end{aligned}$$

Since $(1 + \beta\gamma^2)^{-1/2}\gamma$ is monotone increasing for $\gamma > 0$, (3.3) and (3.4) assure

$$(3.7) \quad \tau \| A_{h,\beta} \|^{1/2} \leqslant \tilde{\gamma} < 2 .$$

Therefore applying the following Lemma 4 to the scheme (3.1), we obtain (3.6). This Lemma is an abstract version of the stability criterion used by Fujii [1].

Lemma 4. Let A be a bounded positive selfadjoint operator acting on the Hilbert space X. Let u(t) be the solution of the difference equation

$$D_{\tau\bar{\tau}}u(t) + Au(t) = 0\ , \qquad n\tau\leqslant t<(n+1)\tau,\ n=0,1,2,\cdots,$$

with the initial values

$$u(t) = a^1\ , \qquad 0\leqslant t<\tau$$

$$\qquad = a^1 - \tau a^0,\ -\tau\leqslant t<0.$$

If $\tau\|A\|^{1/2} \leqslant \gamma < 2$, then we have

$$\|A^{1/2}u(t)\|^2 + \|D_{\bar{\tau}}u(t)\|^2$$

$$\leqslant \frac{1+\gamma/2}{1-\gamma/2}\{\|A^{1/2}a^1\|^2 + \|a^0\|^2\}\ , \qquad t\geqslant 0\ .$$

Proof. See the proof of Proposition 3.1 of Ushijima [5].

§4. Error estimate for the approximate solutions.

Consider the closed linear operator $\mathbb{A}$ acting in $\mathbb{X}$ defined by

$$\mathbb{A} = \begin{pmatrix} 0 & -1 \\ A & 0 \end{pmatrix}$$

with the domain

$$D(\mathbb{A}) = \begin{matrix} D(A) \\ \times \\ V \end{matrix}\ .$$

Then $-\mathbb{A}$ generates the semigroup of linear operator $e^{-t\mathbb{A}}$ in $\mathbb{X}$. It is noted that $\mathbb{A}^{-1}$ and $\mathbb{A}_{h,\beta}{}^{-1}$ exist with the norm bounded uniformly in h.

Proposition 5. There is a constant C independent of β,τ and h satisfying

$$\|\mathbb{A}_{h,\beta}{}^{-1}\mathbb{P}_h a - \mathbb{P}_h\mathbb{A}^{-1}a\|_{h,\beta} \leqslant C(\beta\tau + h)\|a\|_{\mathbb{X}}\ .$$

Proof. For $a = \begin{pmatrix} a^1 \\ a^0 \end{pmatrix}$, let $u = \begin{pmatrix} u^1 \\ u^0 \end{pmatrix} = \mathbb{A}^{-1}a$, $u_{h,\beta} = \begin{pmatrix} u^1_{h,\beta} \\ u^0_{h,\beta} \end{pmatrix} = \mathbb{A}_{h,\beta}{}^{-1}a.$

Then we have

$$-u^0 = a^1\ ,\ Au^1 = a^0$$

and

$$-u^0_{h,\beta} = P_{1h}a^1\ , \quad (1+\beta\gamma^2A_h)^{-1}A_hu^1_{h,\beta} = P_{0h}a^0\ .$$

We use auxiliarly $u^1_h = P_{1h}u^1$, which satisfies

$$A_hu^1_h = P_{0h}a^0\ .$$

Therefore

$$u^1_{h,\beta} = (1+\beta\gamma^2A_h)u^1_h\ ,$$

which implies

$$u^1_{h,\beta} - u^1_h = \beta\tau^2 A_h u^1_h = \beta\tau^2 P_{0h} a^0 .$$

Hence

$$\| A_{h,\beta}{}^{1/2}(u^1_{h,\beta} - P_{1h}u^1)\|$$
$$= \| A_{h,\beta}{}^{1/2}(u^1_{h,\beta} - u^1_h)\|$$
$$= \beta\tau^2 \| A_{h,\beta}{}^{1/2} P_{0h} a_0 \|$$
$$\leqslant \text{ (by (3.7))}$$
$$\leqslant 2\beta\tau \| P_{0h} a_0 \| .$$

So we have

$$\text{(4.1)} \qquad \| u^1_{h,\beta} - P_{1h}u^1 \|_{h,\beta} \leqslant 2\beta\tau \| a_0 \| .$$

From Proposition 4.1 of Ushijima [6], which is an abstract version of Aubin-Nitsche's duality argument, condition (A) implies

$$\text{(4.2)} \qquad \| P_{1h}v - v \| \leqslant Ch \| A^{1/2} v \| , \qquad v \in D(A^{1/2}).$$

On the other hand it holds that

$$\begin{aligned} \| u^0_{h,\beta} - P_{0h}u^0 \| &= \| P_{1h}a^1 - P_{0h}a^1 \| \\ &\leqslant \| P_{1h}a^1 - a^1 \| + \| a^1 - P_{0h}a^1 \| \\ &\leqslant 2 \| P_{1h}a^1 - a^1 \| . \end{aligned}$$

By (4.2), we have

$$\text{(4.3)} \qquad \| u^0_{h,\beta} - P_{0h}u^0 \| \leqslant Ch \| A^{1/2} a^1 \| .$$

Since

$$\| \mathbb{A}_{h,\beta}{}^{-1}\mathbb{P}_h a - \mathbb{P}_h \mathbb{A}^{-1} a \|_{h,\beta}{}^2$$
$$= \| u^1_{h,\beta} - P_{1,h}u^1 \|^2_{h,\beta} + \| u^0_{h,\beta} - P_{0,h}u^0 \|^2 ,$$

(4.2) and (4.3) assure the conclusion of Proposition 5.

<u>Proposition 6.</u> The bounded inverse $(\mathbb{A}_{h,\beta} + \tau\mathbb{B}_{h,\beta})^{-1} \in L(\mathbb{X}_{h,\beta})$ exists and satisfy

$$\| \mathbb{A}_{h,\beta}{}^{-1} - (\mathbb{A}_{h,\beta} + \tau\mathbb{B}_{h,\beta})^{-1} \|_{L(\mathbb{X}_{h,\beta})} \leqslant 3\tau .$$

<u>Proof.</u> In fact we have

$$(\mathbb{A}_{h,\beta} + \tau\mathbb{B}_{h,\beta})^{-1} = \mathbb{A}_{h,\beta}{}^{-1} + \tau\mathbb{K}_{h,\beta}$$

where $\mathbb{K}_{h,\beta} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$.

Since

$$(\mathbb{A}_{h,\beta} + \tau\mathbb{B}_{h,\beta})^{-1}\mathbb{A}_{h,\beta} = 1 + \begin{pmatrix} 0 & 0 \\ \tau A_{h,\beta} & 0 \end{pmatrix} ,$$

we have

$$\| (\mathbb{A}_{h,\beta} + \tau\mathbb{B}_{h,\beta})^{-1}\mathbb{A}_{h,\beta}\|_{L(\mathbb{X}_{h,\beta})} \leqslant 1 + \tau\| A_{h,\beta}{}^{1/2}\| .$$

Hence by (3.7)

$$(4.4) \qquad \| (\mathbb{A}_{h,\beta} + \tau\mathbb{B}_{h,\beta})^{-1}\mathbb{A}_{h,\beta}\|_{L(\mathbb{X}_{h,\beta})} \leqslant 3.$$

And $\mathbb{A}_{h,\beta}{}^{-1}\mathbb{B}_{h,\beta}\mathbb{A}_{h,\beta}{}^{-1} = -\mathbb{K}_{h,\beta}$

implies

$$(4.5) \qquad \| \mathbb{A}_{h,\beta}{}^{-1}\mathbb{B}_{h,\beta}\mathbb{A}_{h,\beta}{}^{-1}\|_{L(\mathbb{X}_{h,\beta})} = 1.$$

Noticing the equality

$$\begin{aligned}
&(\mathbb{A}_{h,\beta} + \tau\mathbb{B}_{h,\beta})^{-1} - \mathbb{A}_{h,\beta}{}^{-1} \\
&= -\tau(\mathbb{A}_{h,\beta} + \tau\mathbb{B}_{h,\beta})^{-1}\mathbb{B}_{h,\beta}\mathbb{A}_{h,\beta}{}^{-1} \\
&= -\tau\{(\mathbb{A}_{h,\beta} + \tau\mathbb{B}_{h,\beta})^{-1}\mathbb{A}_{h,\beta}\}\{\mathbb{A}_{h,\beta}{}^{-1}\mathbb{B}_{h,\beta}\mathbb{A}_{h,\beta}{}^{-1}\} ,
\end{aligned}$$

we have the conclusion from (4.4) and (4.5) .

Propositions 5 and 6 imply the following Proposition 7, which in turn implies Proposition 8 by Theorem 2.

Proposition 7. There is a constant C independent of β,τ and h satisfying

$$\| (\mathbb{A}_{h,\beta} + \tau\mathbb{B}_{h,\beta})^{-1}\mathbb{P}_h \mathbf{a} - \mathbb{P}_h\mathbb{A}^{-1}\mathbf{a}\|_{h,\beta} \leqslant C((1 + \beta)\tau + h)\| \mathbf{a}\|_{\mathbf{X}} .$$

Proposition 8. There is a constant C_T satisfying

$$\| (\mathbb{T}_h(t)P_h - \mathbb{P}_h e^{-t\mathbb{A}})\mathbf{a}\|_{h,\beta} \leqslant C_T(h + \tau)\| \mathbb{A}^2\mathbf{a}\|_{\mathbf{X}}$$

for $0\leqslant t\leqslant T$, $\mathbf{a}\in D(\mathbb{A}^2)$

provided that the stability condition stated in Proposition 3 is satisfied.

Proof of Theorem 1. Let $a^1 \in D(A^{3/2})$ and $a^0 \in D(A)$. This condition is equivalent to $\mathbf{a} = \binom{a^1}{a^0} \in D(\mathbb{A}^2)$. Since

$$\| \mathbb{A}^2\mathbf{a}\|_{\mathbf{X}}{}^2 = \| A^{3/2}a^1 \|^2 + \| Aa^0\|^2 .$$

Let $\mathbf{u}(t) = e^{-t\mathbb{A}}\mathbf{a}$. Then $\mathbf{u}(t) = \begin{pmatrix} u(t) \\ \frac{d}{dt}u(t) \end{pmatrix}$ where $u(t)$ is the solution of the continuous problem (E). Similarly let $\mathbf{u}_h(t) = \mathbf{T}_h(t)\mathbb{P}_h\mathbf{a}$.
Then $\mathbf{u}_h(t) = \begin{pmatrix} u_h(t) \\ D_{\bar\tau}u_h(t) \end{pmatrix}$ where $u_h(t)$ is the solution of $(E^{\tau}_{h,\beta})$.

Now we have

$$(4.6) \quad \mathbf{u}_h(t) - \mathbf{u}(t) = (\mathbf{u}_h(t) - \mathbb{P}_h\mathbf{u}(t)) + (\mathbb{P}_h\mathbf{u}(t) - \mathbf{u}(t)) .$$

It is easy to see

$$(4.7) \quad \| P_h\mathbf{u}(t) - \mathbf{u}(t)\|_{\mathbf{X}} \leq Ch\|\mathbb{A}\mathbf{a}\|_{\mathbf{X}} , \quad t\geq 0 .$$

As for the 1st term of the right-hand side of (4.6), it is noted that

$$(4.8) \quad \|\mathbf{u}_h(t) - \mathbb{P}_h\mathbf{u}(t)\|^2_{\mathbf{X}} = \|\mathbf{u}_h(t) - \mathbb{P}_h\mathbf{u}(t)\|^2_{h,\beta} + \beta\tau^2\| A_{h,\beta}{}^{1/2}A_h{}^{1/2}(u_h(t) - P_{1h}u(t))\|^2$$

Proposition 8 implies that

$$(4.9) \quad \|\mathbf{u}_h(t) - \mathbb{P}_h\mathbf{u}(t)\|^2_{h,\beta} \leq C_T(h^2 + \tau^2)\|\mathbb{A}^2\mathbf{a}\|_{\mathbf{X}} , \quad 0\leq t\leq T.$$

The remaining thing is to obtain the following estimate:

$$(4.10) \quad \| A_{h,\beta}{}^{1/2}A_h{}^{1/2}(u_h(t) - P_{1h}u(t))\| \leq C_T\|\mathbb{A}^2\mathbf{a}\|_{\mathbf{X}} , \quad 0\leq t\leq T.$$

To do this, we note

$$\| A_{h,\beta}{}^{1/2}A_h{}^{1/2}(u_h - P_{1h}u)\|^2 = (A_{h,\beta}{}^{1/2}(u_h - P_{1h}u), A_{h,\beta}{}^{1/2}(A_hu_h - A_hP_{1h}u))$$

Therefore by Schwartz inequality

$$(4.11) \quad \| A_{h,\beta}{}^{1/2}A_h{}^{1/2}(u_h - P_{1h}u)\|^2 \leq \frac{1}{2}(\| A_{h,\beta}{}^{1/2}(u_h - P_{1h}u)\|^2 + \| A_{h,\beta}{}^{1/2}(A_hu_h - P_{1h}u)\|^2).$$

It holds that

$$\begin{aligned} &\| A_{h,\beta}{}^{1/2}(u_h - P_{1h}u)\| \\ &\leq \|\mathbf{u}_h - \mathbb{P}_h\mathbf{u}\|_{h,\beta} \\ &\leq \|\mathbf{u}_h\|_{h,\beta} + \|\mathbb{P}_h\mathbf{u}\|_{h,\beta} \\ &\leq \|\mathbf{T}_h(t)\mathbf{a}_h\|_{h,\beta} + \|\mathbb{P}_h\mathbf{u}\|_{\mathbf{X}} \\ &\leq \left(\frac{1+\tilde\gamma/2}{1-\tilde\gamma/2}\right)^{1/2}\|\mathbf{a}_h\|_{h,\beta} + \|\mathbf{a}\|_{X} , \end{aligned}$$

since $\|\mathbf{u}(t)\|_{\mathbf{X}} = \|e^{-t\mathbb{A}}\mathbf{a}\|_{\mathbf{X}} = \|\mathbf{a}\|_{\mathbf{X}}$ and $\|\mathbf{a}_h\|_{h,\beta} \leq \|\mathbf{a}_h\|_{\mathbf{X}}$.

Therefore

$$(4.12) \qquad \| A_{h,\beta}{}^{1/2}(u_h - P_{1h}u)\| \leqslant (1 + (\frac{1+\tilde{\gamma}/2}{1-\tilde{\gamma}/2})^{1/2})\| a\|_{\mathbf{X}} .$$

Next we note that

$$(4.13) \qquad \| A_{h,\beta}{}^{1/2}(A_h u_h - P_{1h}u)\| \leqslant \| A_{h,\beta}{}^{1/2}A_h u_h\| + \| A_{h,\beta}{}^{1/2}P_{1h}u\| .$$

It holds that

$$\| A_{h,\beta}{}^{1/2}P_{1h}u\| \leqslant \| A^{1/2}P_{1h}u\| \leqslant \| A^{1/2}u\| \leqslant \| \mathbf{u}\|_{\mathbf{X}}$$

where the second inequality follows from the fact that P_{1h} is the orthogonal projection from V onto V_h. The unitarity of $e^{-t\mathbf{A}}$ implies

$$(4.14) \qquad \| A_{h,\beta}{}^{1/2}P_{1h}u\| \leqslant \| a\|_{\mathbf{X}} .$$

Let $v_h = A_h u_h$. Then v_h is the solution of (3.2) with the initial datum $a_h^1 = A_h P_{1h}a^1$ and $a_h^0 = A_h P_{0h}a^0$. Let $\mathbf{v}_h = \begin{pmatrix} v_h \\ D_{\bar{\tau}}v_h \end{pmatrix}$. Since we have

$$\| A_{h,\beta}{}^{1/2}A_h u_h\| \leqslant \| \mathbf{v}_h\|_{h,\beta} ,$$

Proposition 3 implies

$$(4.15) \qquad \| A_{h,\beta}{}^{1/2}A_h u_h\|^2$$
$$\leqslant \frac{1+\tilde{\gamma}/2}{1-\tilde{\gamma}/2}(\| A_{h,\beta}{}^{1/2}A_h P_{1h}a^1\|^2 + \| A_h P_{0h}a^0\|^2) .$$

It holds that

$$\begin{aligned}
&\| A_{h,\beta}{}^{1/2}A_h P_{1h}a^1\| \\
&\leqslant \| A_h{}^{1/2}A_h P_{1h}a^1\| \\
&= \| A_h{}^{1/2}P_{0h}Aa^1\| \\
&\leqslant \| A_h{}^{1/2}(P_{0h}Aa^1 - P_{1h}Aa^1)\| + \| A_h{}^{1/2}P_{1h}Aa^1\| \\
&\leqslant (\text{ by condition (I) and the orthogonality of } P_{1h}) \\
&\leqslant Ch^{-1}\| P_{0h}Aa^1 - P_{1h}Aa^1\| + \| A^{3/2}a^1\| \\
&\leqslant 2Ch^{-1}\| P_{1h}Aa^1 - Aa^1\| + \| A^{3/2}a^1\| \\
&\leqslant (\text{ by (4.2) }) \\
&\leqslant 2Ch^{-1}\cdot h\| A^{3/2}a^1\| + \| A^{3/2}a^1\| .
\end{aligned}$$

Therefore we have

$$(4.16) \qquad \| A_{h,\beta}{}^{1/2}A_h P_{1h}a^1\| \leqslant C\| \mathbf{A}^2 a\|_{\mathbf{X}} .$$

Analogously as above we have

$$\begin{aligned}&\| A_h P_{0h} a^0 \| \\ &\leqslant \| A_h (P_{0h} a^0 - P_{1h} a^0) \| + \| A_h P_{1h} a^0 \| \\ &\leqslant Ch^{-2} \| P_{1h} a^0 - a^0 \| + \| P_{0h} A a^0 \| \\ &\leqslant (\text{ by the duality argument }) \\ &\leqslant C\, h^{-2} h^2 \| Aa^0 \| + \| Aa^0 \| .\end{aligned}$$

Hence we have

$$(4.17) \qquad \| A_h P_{0h} a^0 \| \leqslant C \| \mathbb{A}^2 \mathbf{a} \|_{\mathbf{X}} .$$

Inserting (4.16) and (4.17) into (4.15), then we obtain

$$(4.18) \qquad \| A_{h,\beta}{}^{1/2} A_h u_h \| \leqslant C \| \mathbb{A}^2 \mathbf{a} \|_{\mathbf{X}} .$$

Again inserting (4.14) and (4.18) into (4.13), we have

$$(4.19) \qquad \| A_{h,\beta}{}^{1/2} (A_h u_h - P_{1h} u) \| \leqslant C \| \mathbb{A}^2 \mathbf{a} \|_{\mathbf{X}} .$$

The estimates (4.11), (4.12) and (4.19) assure (4.10). From (4.6) to (4.10), we have the conclusion of Theorem 1.

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