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CONTENTS

Stability Property

of

the Urabe's Implicit Single-Step Method

and its Modification

By

Taketomo Mitsui

§1. Introduction

We are concerning with the numerical solution for the initial-value problem of ordinary differential equations:

(E) $\frac{dy}{dx} = f(x,y), \quad x_I < x < x_T,$

(IV) $y(x_I) = y_I.$

In 1970 Urabe proposed an implicit single-step method for this problem ([6]). Its feature is the following:

(i) Utilizing the second derivative evaluations for the solution of the differential equation (E).

(ii) Essentially single-step predictor-corrector type method.

(iii) High-order accuracy.

Here the second derivative for the solution means

$$g(x,y) = f_x(x,y) + f_y(x,y)f(x,y),$$

which is provided to give in an analytical form.

Suppose that the approximate solution y_n for $y(x)$ and an appropriate initial guess y_{n+1} for $y(x+h)$ are given. Then the Urabe's method first predicts y_{n+2} approximating $y(x+2h)$, and next corrects y_{n+1} by the informations y_n, y_{n+2}, old y_{n+1}, and their first and second derivatives. Such process is continued until convergence. Moreover it has an automatic step-size control strategy.

Urabe applied the method to some numerical examples, which showed the efficiency of the method. It is remarkable that the method is readily applicable to a system of the equations due to its linearity, though he applied to only scalar equations.

The aim of the present paper is to investigate the stability property of the method, and to modify the method to have a built-in estimator for the local truncation error without reducing its stability.

§2. Original Urabe's Method.

First of all we will state the initial-value problem and its discretization. Find the solution of the differential equation

(2.1) $\frac{dy}{dx} = f(x,y), \quad x_I < x < x_T$

subject to the initial condition

(2.2) $y(x_I) = y_I,$

where $y(x)$ belongs to $C^1([x_I,x_T], \mathbb{R}^d)$, $f(x,y)$ to $C^1([x_I,x_T]\times \mathbb{R}^d, \mathbb{R}^d)$. The second derivative of $y(x)$ is given by

$$g(x,y) = f_x(x,y) + f_y(x,y)\, f(x,y).$$

An equi-distant discretization is employed for the problem on $[x_0, x_0+2h]$. Here h is the step-size, $x_i = x_0+ih$ (i=0, 1, 2) are the step-points, y_i (i=0,1, 2) stand for the approximations for $y(x_i)$, and $f_i = f(x_i,y_i)$, $g_i = g(x_i,y_i)$.

The numerical integration method is described as follows. Given the computed value y_0 and an initial guess y_1, make a predicted value y_2 for $y(x_2)$ by

(2.3) $y_2 = -31y_0 + 32y_1 - h(14f_0 + 16f_1) + h^2(-2g_0 + 4g_1).$

Note that (2.3) is an explicit formula for y_2. Then compute the corrected value $\tilde{y}_1$ by

$$(2.4)\quad \tilde{y}_1 = y_0 + \frac{h}{240}(101f_0 + 128f_1 + 11f_2) + \frac{h^2}{240}(13g_0 - 40g_1 - 3g_2).$$

The correction should be continued until the estimate

$$(2.5)\quad \|\tilde{y}_1 - y_1\| < \alpha$$

holds for a previously given criterion α of convergence. If the estimate is not satisfied, replace y_1 by $\tilde{y}_1$ and repeat the process.

The method has some devices attached to it for implementation. For the details, see [6].

(a) Step-size control strategy. The step-size h should be chosen so as to guarantee the convergence of inner iteration. It is derived that h must satisfy

$$(2.6)\quad 2|h|\cdot\|f_y(x_0,y_0)\| < k$$

for a positive constant $k<1$. If not, h should be halved until it satisfies (2.6). If twice of h satisfies the condition, then h should be doubled to enhance the efficiency of the integration. Since the method is essentially a single-step process, the step-size changing causes no trouble except the initial guess for y_1 (see (c) below).

(b) Relations between α and k. Denoting the bound of the round-off error of the machine by ε, the correction to convergence gives the final guess y_1 such as

$$(2.7)\quad \|y_1 - y(x_1)\| \leqq \varepsilon$$

while ε and k satisfy the relation

$$\frac{\varepsilon}{k} \geqq \alpha \geqq \frac{\varepsilon}{1-k}$$

and the estimates (2.5), (2.6) hold. The choice $\alpha = (1/2)\{(\varepsilon/k) + (\varepsilon/(1-k))\}$ seems to be suitable.

(c) Initial guess for y_1. If the step-size has changed at the current step, we may take $y_0 + hf_0 + (1/2)h^2 g_0$ as an initial guess for y_1. Otherwise y_2 at the former step plays it.

On the local accuracy of the method, it can be shown that for the predictor the estimate

$$y(x_2) = -31y(x_0) + 32y(x_1) - h[14f(x_0,y(x_0)) + 16f(x_1,y(x_1)]$$
$$+h^2[-2g(x_0,y(x_0))+4g(x_1,y(x_1))] + \frac{1}{90} h^6 y^{(6)}(x_0) + O(h^7)$$

holds and for the corrector

$$y(x_1) = y(x_0)+ \frac{h}{240}[101f(x_0,y(x_0))+128f(x_1,y(x_1))+11f(x_2,y(x_2))]$$
$$+ \frac{h^2}{240} [13g(x_0,y(x_0)) - 40g(x_1,y(x_1)) - 3g(x_2,y(x_2))]$$
$$+ \frac{1}{9450} h^7 y^{(7)}(x_0) + O(h^8).$$

Hence the corrector has the order 6 whereas the predictor has the order 5. Let us call the terms

$$\frac{1}{90} h^6 y^{(6)}(x_0) \quad \text{and} \quad \frac{1}{9450} h^7 y^{(7)}(x_0)$$

in the above equations the principal parts of the local truncation error for the predictor and the corrector respectively.

As for the convergence of the method the following statement has been established. Assume that $f(x,y)$ and $g(x,y)$ satisfy the Lipschitz condition with respect to y, and that the step-size h (strictly speaking it depends on x) is bounded by $H>0$, i.e. $|h| < H$. Then the integration method is convergent as $H \to 0$ provided that the initial-value of the integration at x_I is identical to y_I.

Remark. The predictor and the corrector equations (2.3), (2.4) are linear with respect to f and g . Hence the method is feasible to apply the system of equations.

§3. Stability Analysis for the Urabe's Method.

Our goal in the present section is

Theorem 1. The Urabe's method is A-stable for a fixed step-size h.

J. R. Cash makes the following comments in his monograph [2]. "Also these schemes (= Urabe's method) do not seem to possess sufficiently large regions of absolute stability for them to be of any use in integrating stiff systems of equations."

But we can obtain the Theorem 1.

Proof of Theorem 1. Employing the scalar test equation

$$(3.1)\qquad \frac{dy}{dx} = \lambda y, \qquad \lambda \in \mathbb{C}, \quad \mathrm{Re}\,\lambda < 0,$$

we have

$$(3.2)\qquad y_2 = h_0(z)y_0 + h_1(z)y_1 \qquad (z = \lambda h)$$

for the predicting process, where

$$h_0(z) = -31 - 14z - 2z^2, \qquad h_1(z) = 32 - 16z + 4z^2.$$

Then the correcting process gives

$$(3.3)\qquad \tilde{y}_1 = c_0(z)y_0 + c_1(z)y_1 + c_2(z)y_2$$

$$= \{c_0(z) + c_2(z)h_0(z)\}y_0 + \{c_1(z) + c_2(z)h_1(z)\}y_1.$$

Here

$$c_0(z) = 1 + \frac{101}{240}z + \frac{13}{240}z^2, \qquad c_1(z) = \frac{128}{240}z - \frac{40}{240}z^2,$$

$$c_2(z) = \frac{11}{240}z - \frac{3}{240}z^2.$$

Hence we can see that for the above linear test equation the method is equivalent to solve the equation

$$(3.4)\qquad \{1 - c_1(z) - c_2(z)h_1(z)\}y_1 = \{c_0(z) + c_2(z)\, h_0(z)\}y_0$$

with respect to y_1. Thus we may call the rational function

$$P(z) = \frac{c_0(z) + c_2(z)\, h_0(z)}{1 - c_1(z) - c_2(z)\, h_1(z)}$$

the stability factor of the method.

The region of absolute stability, say $\mathcal{R}$, is defined by

$$\mathcal{R} = \{z \in \mathbf{C};\ \ \mathrm{Re}\, z < 0,\ \ |P(z)| \leq 1\}$$

for the method. By some manipulations we have

$$(3.5)\qquad P(z) = \frac{3z^4 + 10z^3 - 24z^2 - 120z + 120}{2(3z^4 - 23z^3 + 78z^2 - 120z + 60)}\ .$$

Hence, defining the function $F(x,y)$ by

$$F(x+iy) = |\text{denominator of } P(x+iy)|^2 - |\text{numerator of } P(x+iy)|^2,$$

we have the identity

$$(3.6)\qquad F(x,y) = 27y^8+36y^6x(3x-17)+6y^4x(27x^3-306x^2+672x-1232)$$
$$+12y^2x(9x^5-153x^4+672x^3-1952x^2+3600x-3360)$$
$$+3x(9x^7-204x^6+1344x^5-5344x^4+16320x^3-31360x^2+28800x-9600),$$

which is a polynomial of y^2 and each coefficient of powers of y^2 is again a polynomial of x. Moreover each polynomial of x has positive coefficients for even powers of x and negative coefficients for odd power of x. Thus $F(-x,y) > 0$ holds for any $x > 0$ and any y. Hence $\mathcal{R}$ includes the left half-plane of C. This is the desired result. ∎

By (3.5), we see $|P(z)| \to 1/2$ as $|z| \to \infty$, Rez<0. This fact suggests that the method is not L-stable, but the rate of convergence to zero of the solution for (3.1) by the method is not so much slow while $|z|$ is sufficiently large and Rez<0.

§4. A Modified Method

As is seen in §2, the orders of the principal part of local truncation error do not balance between the corrector and the predictor in the prescribed method. To establish a Milne-type estimator for the local truncation error, however, it is desired to derive a balanced order method. But balancing the order of the predictor with the original corrector is impossible. Thus we will reduce the order of the corrector by one to balance it with that of the predictor.

After the derivation in the original work we easily have the following alternative for the corrector equation:

$$(4.1) \qquad \tilde{y}_1 = y_0 + \frac{h}{240}(110f_0+128f_1+2f_2) + \frac{h^2}{60}(4g_0-7g_1),$$

which has the principal part of the local truncation error as

$$T = -\frac{1}{2400} h^6 y^{(6)}(x_0).$$

The equation (4.1) has been already given by J. R. Cash [1].

We will employ (2.3) as the predictor for y_2 and (4.1) as the corrector for y_1. Then the step-size control strategy is subject to the inequality

$$(4.2) \qquad \frac{192}{240}|h|\cdot\|f_y(x_0,y_0)\| < k$$

instead of (2.6). Other implementing devices (b), (c) in §2 are attached to the new integration method.

A-posteriori estimation of the local truncation error is derived by the following way. Suppose that the step-size h is invariant at the current step. According to the device (c) the initial guess $y_1^{[0]}$ for y_1 is identical with the former y_2. Hence the dominant truncation error is given by

$$(4.3) \qquad y(x_1) - y_1^{[0]} = \frac{1}{90} h^6 y^{(6)}(x_0 - \theta h), \quad 0 < \theta < 1$$

where θ is a constant depending on $y_1^{[0]}$. On the other hand the correction to convergence after m-times iteration yields

$$(4.4) \qquad y(x_1) - y_1^{[m]} = -\frac{1}{2400} h^6 y^{(6)}(x_0 + \tilde{\theta} h), \quad 0 < \tilde{\theta} < 1$$

where $\tilde{\theta}$ is a constant depending on $y_1^{[m]}$. Assume that

$y^{(6)}(x_0 - \theta h) \fallingdotseq y^{(6)}(x_0 + \tilde{\theta} h)$ which is valid for a sufficiently small positive h. Then we have from (4.3) and (4.4)

$$(4.5) \qquad y_1^{[m]} - y_1^{[0]} = \frac{83}{7200} h^6 y^{(6)}(x_0).$$

Substituting (4.5) into (4.4), we obtain

$$(4.6) \qquad y(x_1) - y_1^{[m]} = -\frac{3}{83} e_m ,$$

where $e_m = y_1^{[m]} - y_1^{[0]}$. Thus we can establish an a-posteriori estimate (4.6) for the local truncation error.

Ordinarily there are two ways of usage for the a-posteriori estimate (4.6). One is to apply it to the step-size control so that the integration may proceed in maintaining the local truncation error within the bound of previously given tolerance. But this way spoils the basis (4.3) of the a-posteriori estimate for the next step, because it depends on the invariantness of the step-size. The other is to modify the value $y_1^{[m]}$ to obtain possibly more accurate approximate value by it. That is, after the iteration to convergence we employ the modifying process

$$(4.7) \qquad y_1^* = y_1^{[m]} - \frac{3}{83} e_m.$$

It is remarkable that as is ordinary for the modifier in the PC methods the process (4.7) does not always give more accurate value because the identity (4.5) holds merely in an approximate sense.

Concluding the present section, we will mention some features of the modified method (with or without the modifier). Though the local accuracy is reduced than that of the original method, it has two advantages. One is the reduction of the number of function evaluations per step. As is seen in (4.1), the corrector needs two evaluations for g while the original corrector (2.4) needs three evaluations. The other is the extension of the allowable step-size. The estimate (4.2) ensures almost twice for the possible step-size against that by the original (2.6). These features will be expected to make the modified method more efficient.

§5. Stability Analysis for the Modified Method

In the present section we will show that the stability for the modified method is not reduced in comparison with those for the original method. Corresponding to Th. 1 we have the following theorem.

<u>Theorem 2</u>. The modified method without the modifier is A-stable for a fixed step-size h.

<u>Proof</u>. Similar to the original method, the scalar test equation (3.1) is employed. Since the predicting process is same as in the original case, we see

$$y = h_0(z)y_0 + h_1(z)y_1 \qquad (z = \lambda h).$$

Then the correcting process (4.1) implies

$$(5.1) \qquad y_1 = d_0(z)y_0 + d_1(z)y_1 + d_2(z)y_2$$
$$= \{d_0(z) + d_2(z)h_0(z)\}y_0 + \{d_1(z) + d_2(z)h_1(z)\}y_1.$$

Here

$$d_0(z) = 1 + \frac{101}{240} z + \frac{4}{60} z^2 , \quad d_1(z) = \frac{128}{240} z - \frac{7}{60} z^2, \quad d_2(z) = \frac{2}{240} z.$$

Hence instead of (3.4) we have

$$(5.2) \qquad \{1-d_1(z)-d_2(z)h_1(z)\}y_1 = \{d_0(z)+d_2(z)h_0(z)\}y_0.$$

The stability factor Q(z) of the modified method is given by

$$(5.3) \qquad Q(z) = \frac{d_0(z) + d_2(z)\, h_0(z)}{1 - d_1(z) - d_2(z)\, h_1(z)}$$

$$= \frac{z^3 + 3z^2 - 12z - 60}{2z^3 - 15z^2 + 48z - 60} .$$

Thus the region of absolute stability for the modified method may be defined by

$$\mathcal{R}_M = \{z \in \mathbf{C};\ \mathrm{Re}z<0,\ |Q(z)| \leqq 1\}.$$

Similar to the proof of Th. 1, the function

$$G(x,y) = |\text{denominator of } Q(x+iy)|^2 - |\text{numerator of } Q(x+iy)|^2$$

is shown to be positive for any $x<0$ and any y, because the identity

$$G(x,y) = 3\{y^6+y^4x(3x-22)+y^2x(3x^3-44x^2+144x-3360)$$
$$+x(x^5-22x^4+144x^3-496x^2+1440x-2400)\}$$

holds. Hence $\mathcal{R}_M$ includes the left half-plane of $\mathbf{C}$. ∎

By virtue of (5.3), we can see $|Q(z)| \to 1/2$ as $|z| \to \infty$, $\mathrm{Re}z<0$. Hence the comments on the L-stability made in the last paragraph in §3 is also valid to this case.

§6. Numerical Examples

To illustrate the efficiency of the modified method with the modifier (4.7) we will show some numerical examples whose calculations were carried out by FORTRAN on DEC System2020 in the Research Institute for Mathematical Sciences.

Example 1. Scalar case.

This is the problem appeared in the original Urabe's paper. The differential equation is

$$\frac{dy}{dx} = 5x\left(\frac{1}{2} - y\right)^{4/5}, \quad -1 < x < 1,$$

the initial condition is $y(-1) = 15/32$. The problem has the exact solution $y^*(x) = (1/2) - (1 - x^2/2)^5$. We integrate it both by the original method and the modified one (abbreviated by URABE and URABEM, respectively) for their comparison. The computational constants are the followings;

the machine epsilon = 3.73×10^{-9},

the contraction constant $k = 0.5$,

the convergence criterion $\alpha = 1.1 \times 10^{-8}$,

the maximum allowable step-size $H = 2^{-4}$.

The result is given in Table 1.

Table 1.

method	absolute error at x=1	step numbers	number of function evaluations f	g
original	1.1×10^{-8}	42	416	416
modified	1.2×10^{-7}	32	220	126

Example 2. The orbit equations.

Famous orbit equations are integrated numerically. They are a system of four components (y_1, y_2, y_3, y_4) with the independent variable x, $0 < x < 2$.

$$\frac{dy_1}{dx} = \pi y_3, \quad \frac{dy_2}{dx} = \pi y_4, \quad \frac{dy_3}{dx} = \frac{-\pi y_1}{(y_1^2 + y_2^2)^{1/2}}, \quad \frac{dy_4}{dx} = \frac{-\pi y_2}{(y_1^2 + y_2^2)^{1/2}}$$

The initial conditions are

$$y_1(0) = 1 - e, \quad y_2(0) = y_3(0) = 0, \quad y_4(0) = \sqrt{(1 + e)/(1 - e)},$$

where e is a parameter satifying $0 < e < 1$. We take the case $e=0.1$. The exact solution $y^*(x)$ of the problem is known to be given by

$$y_1(x) = \cos u(x) - e, \quad y_2(x) = (1 - e^2)^{1/2} \sin u(x),$$

$$y_3(x) = -\sin u(x)/(1 - e \cos u(x)),$$

$$y_4(x) = (1 - e^2)^{1/2} \cos u(x)/(1 - e \cos u(x)),$$

where $u(x)$ is an implicitly defined function of x such that

$$u(x) - e \sin u(x) - x = 0.$$

Thus the exact solution has a periodicity as $y_i^*(0) = y_i^*(2)$.

The computational constants are

the contraction constant $k = 0.1$,

the convergence criterion $\alpha = 2.28 \times 10^{-8}$.

Other constants are same as in the Ex. 1. The result is shown in Table 2.

Table 2.

method	absolute error $\|y^*(2) - y(2)\|_\infty$	step numbers	number of function evaluations f	g
URABE	7.7×10^{-7}	257	1515	1515
URABEM	5.0×10^{-7}	99	601	350

Example 3. Euler equation of motion for a rigid body without external forces.

This is the problem to find the solution (y_1, y_2, y_3) for the system of nonlinear differential equations

$$y_1' = y_2 y_3, \quad y_2' = y_1 y_3, \quad y_3' = -0.51 y_1 y_2, \qquad x>0$$

with the initial conditions

$$y_1(0) = 0, \; y_2(0) = y_3(0) = 1.$$

The exact solution is expressed by the Jacobian elliptic functions. In fact, the identities

$$y_1(x) = sn(x, 0.51), \; y_2(x) = cn(x, 0.51), \; y_3(x) = dn(x, 0.51)$$

hold. Hence they have the following periodicities:

$$y_1(x+4K) = y_1(x), \; y_2(x+4K) = y_2(x), \; y_3(x+4K) = y_3(x),$$

$$y_1(K) = 1, \; y_2(K) = 0, \; y_3(K) = 0.7,$$

where the constants K is given by the complete elliptic integral

$$K = \int_0^1 \frac{dx}{\{(1 - x^2)(1 - 0.51x^2)\}^{1/2}} .$$

The numerical integration is done by the following four methods to compare their results.

RKF45 --- Runge-Kutta-Fehlberg method ([3]).

SHAMP --- Shampine-Gordon's VSVO method ([5]).

URABE and URABEM.

Each method integrates the problem and outputs the results in every K interval until x=28K. Computational constants in the double precision arithmetic are the followings;

the contraction constant k = 0.1,

the machine epsilon = 3.73×10^{-8},

the convergence criterion $\alpha = 2.28 \times 10^{-8}$,

the maximum allowable step-size H = 2^{-4},

the tolerance of absolute error in RKF45 and SHAMP = 1.0×10^{-8},

the tolerance of relative error in RKF45 and SHAMP = 1.0×10^{-8},

The results are shown in Table 3. Here we adopt the following notations. The absolute error means the magnitude of the difference between numerical and the exact values at x=4nK (n=1, 2, ..., 7) in $\mathbf{R}^3$ equipped with l-2 norm. The relative error means the ratio of the absolute error to the exact value.

The sixth column shows the total CPU times for integration, but they have only a relative account because the computations are carried out under the TSS environment.

Table 3.

method	maximum of abs. error	maximum of rel. error	number of function evaluations f	g	total CPU times
RKF45	7.86×10^{-6}	5.56×10^{-6}	2927	0	9.837 sec
SHAMP	3.75×10^{-7}	2.65×10^{-7}	2072	0	24.876
URABE	3.87×10^{-7}	2.73×10^{-7}	16016	16016	76.462
URABEM	1.42×10^{-6}	1.01×10^{-6}	4606	2730	23.295

Example 4. Stiff problem.

Though URABE and URABEM are both A-stable, they have a significant restriction to apply the stiff problems, which possess a large Lipschitz constant. In this case the estimation (2.6) or (4.2) restricts the allowable step-size to be very small. To illustrate these phenomena we integrate the differential equations

$$\frac{dy_1}{dx} = 0.01 - (0.01 + y_1 + y_2)(y_1^2 + 1001y_1 + 1001),$$

$$\frac{dy_2}{dx} = 0.01 - (0.01 + y_1 + y_2)(1 + y_2^2)$$

under the initial conditions $y_1(0) = 0$, $y_2(0) = 0$ ([4]). The stiffness ratio is known to be equal to 1.012×10^5 at x=0, and 2.438×10^2 at x=100. URABE and URABEM without the modifier (we have established A-stability only for URABEM without the modifier!) spend too many steps even to reach x=0.5. The results are given in Table 4. Here the contraction constant k is taken as 0.9 so that it may allow the step-size as large as possible. Hence the convergence criterion α is equal to 3.937×10^{-8}.

Table 4.A. Computed values

x	URABE y_1 y_2	URABEM y_1 y_2
0.03125	-1.0281691E-02	-1.0281688E-02
	3.0186012E-04	3.0186013E-04
0.06250	-1.0593549E-02	-1.0593542E-02
	6.1372972E-04	6.1372983E-04
0.09375	-1.0905408E-02	-1.0905406E-02
	9.2559904E-04	9.2559935E-04
0.12500	-1.1217282E-02	-1.1217269E-02
	1.2374679E-03	1.2374687E-03
0.15625	-1.1529144E-02	-1.1529140E-02
	1.5493366E-03	1.5493377E-03
0.18750	-1.1841011E-02	-1.1841014E-02
	1.8612051E-03	1.8612066E-03
0.21875	-1.2152863E-02	-1.2152857E-02
	2.1730727E-03	2.1730754E-03
0.25000	-1.2464720E-02	-1.2464727E-02
	2.4849397E-03	2.4849441E-03
0.28125	-1.2776585E-02	-1.2776596E-02
	2.7968065E-03	2.7968123E-03
0.31250	-1.3088456E-02	-1.3088457E-02
	3.1086732E-03	3.1086802E-03
0.34375	-1.3400314E-02	-1.3400330E-02
	3.4205397E-03	3.4205480E-03
0.37500	-1.3712175E-02	-1.3712173E-02
	3.7324061E-03	3.7324154E-03
0.40625	-1.4024024E-02	-1.4024040E-02
	4.0442725E-03	4.0442824E-03
0.43750	-1.4335895E-02	-1.4335909E-02
	4.3561389E-03	4.3561488E-03
0.46875	-1.4647752E-02	-1.4647765E-02
	4.6680053E-03	4.6680152E-03
0.50000	-1.4959604E-02	-1.4959625E-02
	4.9798717E-03	4.9798816E-03

Table 4.B Number of function evaluations.

	URABE	URABEM
For the first derivative f	33418	12238
For the second derivative g	33418	6631
For the Jacobi matrix f_y	4096	1024

§7. Discussions.

The numerical examples in the previous section show the efficiency of our modified method in comparison with the original Urabe's method. It always reduces the number of function evaluations to almost half or less of those by the original method. As for the significant features of the numerical solution it is inferior to the original method by almost one digits as is expected from the theoretical consideration. But, in some cases, for instance in Ex. 2, our modified method gives a slightly more accurate result than the original one. It seems that the modifier works well in this case.

Comparing with the other well-used methods, the original and the modified methods however seem to have not so much advantage. But when we can obtain the analytical form for the second derivative (this is the case by applying the symbolic and algebraic manipulation software), they may be one of the practical ways for the numerical integration.

For the stiff problems the original and the modified (without the modifier) are freed from the instability phenomena. It remains however a problem to remove the restriction for the allowable step-size in the methods to be more efficient for the stiff problems. In such cases two features of the modified method mentioned in the last paragraph in §4 are also expected to make it advantageous to the original one.

References

[1] Cash, J.R., High order methods for the numerical integration of ordinary differential equations, Numer. Math., 30(1978), 385-409.

[2] Cash, J.R., Stable Recursions with Applications to the Numerical

Solution of Stiff Systems, Academic Press, London, 1979.

[3] Forsythe, G.E., Malcolm, M.A. & Moler, C.B., Computer Methods for Mathematical Computations, Prentice-Hall, Englewood Cliffs, 1979.

[4] Lambert, J.D., Computational Methods in Ordinary Differential Equations, John Wiley & Sons, London, 1973.

[5] Shampine, L.F. & Gordon, M.K., Computer Solution of Ordinary Differential Equations, W.H. Freeman & Co., San Francisco, 1975.

[6] Urabe, M., An implicit one-step method of high-order accuracy for the numerical integration of ordinary differential equations, Numer. Math., 15(1970), 151-164.

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The Relation of Certain Numerical Methods for Ill-Posed Problems.

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1. Introduction

In this paper we deal with a linear operator equation

$$K\ f = g$$

which leads to an ill-conditioned linear system, where K is a compact operator of a Hilbert space X into another Hilbert space Y. We are to find $f \in X$ for given $g \in Y$. A well-known example of such a problem is the Fredholm integral equation of the first kind of the form

$$\int_a^b k(s,t)\ f(t)\ dt = g(s), \quad s \in [a,b], \qquad (1)$$

where k(s,t) is an L_2 kernel and f,g are L_2 functions.

Let ϕ_i and ψ_i be singular functions and λ_i ($\lambda_1 \geqq \lambda_2 \geqq \cdots$) be singular values of K, namely $K\ \psi_i = \lambda_i\ \phi_i$ and $K^* \phi_i = \lambda_i\ \psi_i$ for i = 1,2,.... Then it is well known that the unique solution to (1) exists and can be written as

$$f = \sum_{i=1}^{\infty} \frac{1}{\lambda_i} (g,\ \phi_i)\ \psi_i \quad , \qquad (2)$$

if and only if g belongs to the range of K and

$$\sum_{i=1}^{\infty} \frac{1}{\lambda_i} (g, \phi_i)$$

converges. The sequence $\{\lambda_i\}$ then converges to zero as $i \to \infty$. The degree of ill-conditioning for (1) depends on the rate at which the sequence of the singular values goes to zero. This is discussed by Smithies [3] and Weyl [5]. Their theorems imply that the rate of the convergence, which provides a measure of conditioning, depends on the smoothness of the kernel $k(s,t)$. When the kernel is analytic and smooth, the sequence $\{\lambda_i\}$ may go to zero exponentially. In this case any direct approaches to solve the equation (1) in the ordinally least squares sense may produce disastrous results due to the numerical instability.

To cope with this difficulty, several numerical methods have been developed. Best known of them are the method of truncated singular value and the method of regularization. In this paper we discuss above two methods under the assumption that the sequence of the singular values goes to zero exponentially. It is shown that the both methods can be identified by introducing a variable transformation for the regularization parameter.

Throughout this paper we assume that $X = Y = L_2(a,b)$ and each set of the singular functions $\{\phi_i\}$ and $\{\psi_i\}$ forms an orthonormal system. Further we suppose that the sequence $\{\lambda_i\}$ converges to zero exponentially. This is the case when the kernel is smooth and the resulting linear system is severely ill-conditioned.

2. The method of truncated singular value

Let S_n be Span $\{\psi_1, \psi_2, \ldots, \psi_n\}$ which is a subspace of X. Then the method of truncated singular value finds the function f_t such that

$$\inf_{f \in S_n} \| K f - g \|^2 = \| K f_t - g \|^2 .$$

In terms of the singular functions and the singular values, f_t is given by a truncated series of (2)

$$f_t = \sum_{i=1}^{n} \frac{1}{\lambda_i} (g, \phi_i) \psi_i . \quad (3)$$

An appropriate choice of the subspace S_n is critical to this method. This choice can be actually made by introducing a threshold τ. If corresponding singular value to the singular function ψ_i is smaller than the threshold τ, we do not employ ψ_i to form the subspace S_n.

3. The method of regularization

The solution f_r by the method of regularization is given as a function f_r such that

$$\inf_{f \in X} [\| K f - g \|^2 + \mu \|f\|^2] = \| K f_r - g \|^2 + \mu \| f_r \|^2 ,$$

in which we call $\mu \geq 0$ the regularization parameter. The unique solution f_r is also the solution of the normal equation

$$(K^* K + \mu I) f = K^* g . \quad (4)$$

Thus the solution f_r is given by $f_r = (K^*K + \mu I)^{-1} K^*g$. Using the singular values and the singular functions, f_r can be written as

$$f_r = \sum_{i=1}^{\infty} \frac{\lambda_i}{\lambda_i^2 + \mu} (g, \phi_i) \psi_i \; . \qquad (5)$$

The selection of μ is again critical to this method. The relation between the threshold τ and the regularization parameter μ shall be clarified to some extent in the next section.

4. The relation between the two methods

In this section we deal with a class of ill-conditioned linear systems which satisfies

$$\frac{\lambda_{i+1}}{\lambda_i} < \beta^{-1} \; , \; \beta >> 1 \; , \quad i = 1,2,\dots .$$

Let c_t^i and c_r^i be the coefficients of ψ_i for (3) and (5) respectively, i.e.

$$c_t^i = \begin{cases} \frac{1}{\lambda_i} (g, \phi_i) & \text{for } i \leqq n, \\ 0 & \text{for } i > n, \end{cases}$$

$$c_r^i = \frac{\lambda_i}{\lambda_i^2 + \mu} (g, \phi_i) \qquad \text{for } i = 1,2,\dots .$$

Now suppose that $\lambda_m^2 > \mu \leqq \lambda_{m+1}^2$ for some m. We introduce a variable transformation ν for the regularization parameter μ,

$$\nu = -\log_\beta \mu . \qquad (6)$$

Then we have

$$\frac{\beta^{-\nu}}{\lambda_i^2} \ll 1 \quad \text{for} \quad i = 1,2,\ldots,m-1 \quad ,$$

since

$$\frac{\beta^{-\nu}}{\lambda_{m-1}^2} < \frac{\lambda_m^2}{\lambda_{m-1}^2} < \beta^{-2} \ll 1 .$$

Therefor

$$c_r^i = \frac{\lambda_i}{\lambda_i^2 + \mu} (g, \phi_i)$$

$$= \frac{\frac{1}{\lambda_i}}{1 + \frac{\beta^{-\nu}}{\lambda_i^2}} (g, \phi_i)$$

$$= \frac{1}{\lambda_i} (g, \phi_i) \quad \text{for} \quad i = 1,2,\ldots,m-1. \qquad (7)$$

Similarly we have

$$\frac{\beta^{-\nu}}{\lambda_i^2} \gg 1 \quad \text{for} \quad i = m+2, m+3, \ldots .$$

Let $R^i(\mu)$ be the ratio of c_r^i to $1/\lambda_i \, (g, \phi_i)$. Then

$$R^i(\mu) = \frac{\lambda_i^2}{\lambda_i^2 + \mu}$$

$$= \frac{1}{1 + \frac{\beta^{-\nu}}{\lambda_i^2}}$$

$$\simeq \quad 0 \qquad \text{for} \quad i = m+2, m+3, \ldots. \qquad (8)$$

Further the derivatives of R^i with respect to ν are

$$\frac{dR^i}{d\nu} = \log_e \beta \, \frac{\lambda_i^2}{(\lambda_i^2 + \mu)^2} \, \beta^{-\nu} , \qquad (9)$$

and

$$\frac{d^2R^i}{d\nu^2} = - (\log_e \beta)^2 \, \frac{\lambda_i^2 (\lambda_i^2 - \mu)}{(\lambda_i^2 + \mu)^3} \, \mu . \qquad (10)$$

Thus $\frac{dR^i}{d\nu}$ attains its maximum at $\mu = \lambda_i^2$, or $\nu = \alpha_i$ if we set

$$\lambda_i^2 = \beta^{-\alpha_i} , \quad i = 1,2,\ldots.$$

Then it follows from (7) - (10) that $R^i \simeq 1$ for $\nu \geqq \alpha_i+1$ and goes down to zero very rapidly for $\alpha_i-1 < \nu < \alpha_i+1$ at the speed of $O(\beta^{-\nu})$ and then $R^i \simeq 0$ for $\nu \leqq \alpha_i-1$. This implies that we can approximately identify the method of regularization with that of truncated sigular value under the variable transform ν, provided that we choose τ and μ such that $\lambda_n > \tau > \lambda_{n+1}$ and $\lambda_n^2 > \mu > \lambda_{n+1}^2$. Hence the method of regularization with the regularization parameter μ corresponds to the method of truncated singular value with the choice of the threshold $\tau = \sqrt{\mu}$.

References

[1] J.H.Hilgers, On the equivalence of regularization and certain reproducing kernel Hilbert space approaches for solving first kind problems, SIAM J. Numer. Anal., 13 (1976), pp. 172-184.

[2] B.A.Lewis, On the numerical solution of Fredholm integral equations of the 1st kind, JIMA, 16(1975), pp. 207-220.

[3] F.Smithies, The eigenvalues and singular values of integral equations, Proc. London Math. Soc., 43(1937), pp. 255-279.

[4] J.M.Varah, On the numerical solutions of ill-conditioned linear systems with application to ill-posed problems, SIAM J. Numer. Anal., 10(1973),pp. 257-266.

[5] H.Weyl, Das asymptotiche Vereilungestig der Eigenwerte lineare partiller Differentialgleichungen, Math. Ann., 71(1912), pp. 441-479.

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A Posteriori Improvement of Cubic Spline Approximate Solution of Two Point Boundary Value Problem. Difference Method

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Abstract.

We consider the numerical solution of two point boundary value problem by difference method using cubic spline. We obtain an asymptotic expansion of the error which is a posteriori determined with little additional computation. The applications of this expansion to a posteriori improvement of the approximate solution and an adaptive mesh selection strategy (chopping procedure) are discussed. Some numerical results which closely correspond with the predicted theory are given.

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1. Introduction and Description of Method

We consider the following two-point boundary value problem:

(1.1) $$x'(t) = w(t), \qquad 0 \le t \le 1$$

(1.2) $$w'(t) = f(t, x(t), w(t)), \qquad 0 \le t \le 1$$

(1.3) $$a_0 x(0) - b_0 w(0) = c_0$$

(1.4) $$a_1 x(1) + b_1 w(1) = c_1$$

where $f(t, x, w)$ is defined and sufficiently smooth in a region D of (t, x, w)-space intercepted by two hyperplanes $t = 0$ and $t = 1$. Now making use of B-spline

$$Q_{m+1}(t) = (1/m!) \sum_{i=0}^{m+1} (-1)^i \binom{m+1}{i} (x - i)_+^m ,$$

we consider spline functions $x_h(t)$ and $w_h(t)$ of the form

$$x_h(t) = \sum_{i=-3}^{n-1} \alpha_i Q_4(t/h - i) \qquad (nh = 1)$$

$$w_h(t) = \sum_{i=-2}^{n-1} \beta_i Q_3(t/h - i).$$

The above $x_h(t)$ and $w_h(t)$ will be approximate solutions to the problem (1.1)-(1.4) if they satisfy

(1.5) $$x_h'(t) = w_h(t), \qquad 0 \le t \le 1$$

(1.6) $$w_h'(t) = Pf(t, x_h(t), w_h(t)), \quad 0 \le t \le 1$$

(1.7) $$a_0 x_h(0) - b_0 w_h(0) = c_0$$

(1.8) $$a_1 x_h(1) + b_1 w_h(1) = c_1 .$$

Here P is an operator defined by

(1.9) $$(Pg)(t) = \sum_{i=0}^{n} \gamma_i L_i(t)$$

so that

(1.10)
(i) $\Delta^r \gamma_0 = 0, \qquad r \ge 4$

(ii) $(1/6)(\gamma_{i+1} + 4\gamma_i + \gamma_{i-1}) = g_i$

(iii) $\nabla^r \gamma_n = 0$

where Δ and ∇ are forward and backward difference operators, and $L_i(t)$ is a piecewise linear function with the property

$L_i(t_j) = \delta_{i,j}$ ($t_j = jh$). In practical computation, conditions 1.10(i) and (iii) may be rewritten as follows : for $r = 5$,

$$(1.11) \quad \begin{aligned} \gamma_0 + (19/5)\gamma_1 &= (1/30)(181g_1 - 45g_2 + 9g_3 - g_4) \\ \gamma_n + (19/5)\gamma_{n-1} &= (1/30)(181g_{n-1} - 45g_{n-2} + 9g_{n-3} \\ &\qquad - g_{n-4}) \end{aligned}$$

where $g_i = g(ih)$.

By use of the consistency relation, from 1.10(ii) we have

$$(1.12) \quad \begin{aligned} &(1/h^2)\{ x_h(t_{i+1}) - 2x_h(t_i) + x_h(t_{i-1}) \} \\ &= (1/6)\{ x_h''(t_{i+1}) + 4x_h''(t_i) + x_h''(t_{i-1}) \} \\ &= f(t_i, x_h(t_i), w_h(t_i)), \quad i = 1, 2, \cdots, n - 1 \end{aligned}$$

i.e., if $f(t, x, w) = f(t, x, 0)$ and $b_0 = b_1 = 0$, then the above approximate problem is considered to be the usual central difference scheme with respect to $x_h(t_i)$, $i = 0, 1, \cdots, n$.

By using the similar technique in [5], we have

Theorem 1. If the problem (1.1)-(1.4) has the isolated solution $(\hat{x}, \hat{w})$, there exists a spline solution $(\bar{x}_h, \bar{w}_h)$ of the problem (1.5)-(1.8) of the form

$$(1.13) \quad \bar{x}_h(t) = \sum_{i=-3}^{n-1} \bar{\alpha}_i Q_4(t/h - i)$$

$$(1.14) \quad \bar{w}_h(t) = \sum_{i=-2}^{n-1} \bar{\beta}_i Q_3(t/h - i)$$

so that

$$(1.15) \quad \|\hat{x} - \bar{x}_h\| = \max_t |\hat{x}(t) - \bar{x}_h(t)| = O(h^2)$$

$$(1.16) \quad \|\hat{w} - \bar{w}_h\| = O(h^2).$$

By using this Theorem 1, in the next section we shall prove the following asymptotic expansions of the errors $\hat{x}(t) - \bar{x}_h(t)$ and $\hat{w}(t) - \bar{w}_h(t)$:

Theorem 2. Under the same conditions of Theorem 1, we have at any point $t \in [0, 1]$

(1.17) $\hat{x}(t) - \bar{x}_h(t) = (h^2/12)\phi(t) + O(h^4)$

(1.18) $\hat{w}(t) - \bar{w}_h(t) = (h^2/12)\psi(t) + O(h^3)$

where (ϕ, ψ) is the solution of the following boundary value problem :

(1.19) $\phi'(t) = \psi(t)$

(1.20) $\psi'(t) = f_x(t, \hat{x}(t), \hat{w}(t))\phi(t) + f_w(t, \hat{x}(t), \hat{w}(t))\psi(t) + \hat{x}^{(4)}(t)$

(1.21) $a_0\phi(0) - b_0\psi(0) = 0$

(1.22) $a_1\phi(1) + b_1\psi(1) = 0.$

Corollary. At any mesh point, we have

(1.23) $\hat{x}''(t) - \bar{x}_h''(t) = (h^2/12)\{\phi''(t) + \hat{x}^{(4)}(t)\} + O(h^4).$

By means of this Corollary, we have

(1.24) $\hat{x}^{(4)}(t_i) = (1/h^2)\{ \bar{x}_h''(t_{i+1}) - 2\bar{x}_h''(t_i) + \bar{x}_h''(t_{i-1})\} + O(h^4), i = 1, 2, \cdots, n - 1.$

Let us consider the following approximate problem to (1.19)-(1.22) :

(1.25) $\phi_h'(t) = \psi_h(t)$

(1.26) $\psi_h'(t) = P[\, f_x(t, \bar{x}_h(t), \bar{w}_h(t))\phi_h(t) + f_w(t, \bar{x}_h(t), \bar{w}_h(t))\psi_h(t) + g_h(t) \,]$

(1.27) $a_0\phi_h(0) - b_0\psi_h(0) = 0$

(1.28) $a_1\phi_h(1) + b_1\psi_h(1) = 0$

where $g_h(t)$ is a piecewise linear function and $g_h(t_i) = (1/h^2)\{\bar{x}_h''(t_{i+1}) - 2\bar{x}_h''(t_i) + \bar{x}_h''(t_{i-1})\}$, $i = 1, 2, \cdots, n - 1$.

By applying the same argument in [6] to (1.25)-(1.28), we have

Theorem 3. Under the same conditions of Theorem 1, the problem (1.25)-(1.28) has the solution $(\bar{\phi}_h, \bar{\psi}_h)$ such that

$$(1.29) \qquad \|\phi - \bar{\phi}_h\| , \|\psi - \bar{\psi}_h\| = O(h^2)$$

Since the coefficient matrix of the linear system (1.25)-(1.28) for determining $(\bar{\phi}_h, \bar{\psi}_h)$ is the one of the Newton-method at the final stage by which $(\bar{x}_h, \bar{w}_h)$ is calculated, $(\bar{\phi}_h, \bar{\psi}_h)$ may be determined with very little additional computation. By Theorems 2 and 3, we have asymptotic expansions :

$$(1.30) \qquad \hat{x}(t) - \bar{x}_h(t) = (h^2/12)\bar{\phi}_h(t) + O(h^4)$$

$$(1.31) \qquad \hat{w}(t) - \bar{w}_h(t) = (h^2/12)\bar{\psi}_h(t) + O(h^3).$$

In Section 3 we shall consider chopping procedure applied to the two-point boundary value prblem by using (1.30).

2. Asymptotic Expansions of Errors

Before we proceed with analysis, we shall require the following Lemmas.

Lemma 1. If $\lambda \neq (2 - \sqrt{3})\mu$, then the following n×n tridiagonal matrix A_n is nonsigular and in addition $\|A_n^{-1}\|$ ($\|\cdot\|$ means the maximum matrix norm) is bounded for sufficiently large n

where

$$(2.1) \qquad A_n = \begin{bmatrix} \lambda & \mu & & & \\ 1 & 4 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & 4 & 1 \\ & & & \mu & \lambda \end{bmatrix}.$$

Proof. Let us consider a linear system :

(2.2) $$A_n \xi = \eta$$

where $\xi = (\xi_1, \xi_2, \cdots, \xi_n)$ and $\eta = (\eta_1, \eta_2, \cdots, \eta_n)$.

Hence we have

(2.3) $$\begin{bmatrix} 4 & 1 & & & \\ 1 & 4 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & 4 & 1 \\ & & & 1 & 4 \end{bmatrix} \begin{bmatrix} \xi_2 \\ \xi_3 \\ \vdots \\ \xi_{n-2} \\ \xi_{n-1} \end{bmatrix} = \begin{bmatrix} \eta_2 - \xi_1 \\ \eta_3 \\ \vdots \\ \eta_{n-2} \\ \eta_{n-1} - \xi_n \end{bmatrix}.$$

Let $(a_{i,j}\ ;\ 2 \le i, j \le n-1)$ be the inverse of the coefficient matrix of (2.3), then we have

(2.4) $$\xi_i = \sum_{j=2}^{n-1} a_{i,j}\eta_j - a_{i,2}\xi_1 - a_{i,n-1}\xi_n$$

$$i = 2, 3, \cdots, n-1.$$

By substituting the above relations into the first and last equations of (2.2), we have

(2.5) $$(\lambda - \mu a_{2,2})\xi_1 - \mu a_{2,n-1}\xi_n = \eta_1 - \mu\sum_{j=2}^{n-1} a_{2,j}\eta_j$$

(2.6) $$-\mu a_{n-1,2}\,\xi_1 + (\lambda - \mu a_{n-1,n-1})\xi_n = \eta_n - \mu\sum_{j=2}^{n-1} a_{n-1,j}\eta_j$$

Since $a_{2,2} = a_{n-1,n-1} \to 2 - \sqrt{3}$, $a_{2,n-1} = a_{n-1,2} \to 0$

and $\sum_{i=2}^{n-1} |a_{i,j}| < \frac{1}{2}$, $i = 2, 3, \cdots, n-1$ ([1]),

we have

(2.7) $|\xi_1|$, $|\xi_n| \le C\|\eta\|$ ($\|\cdot\|$ means the maximum vector norm) for a generic constant C independent of n provided that $\lambda \neq (2 - \sqrt{3})\mu$ and n is sufficiently large.

Combining (2.4) and (2.7) yields

(2.8) $\|\xi\| \le C\|\eta\|$ for sufficiently large n

from which follows the desired result.

Since $\Delta^{r+1}\gamma_0 = \Delta^r\gamma_1 - \Delta^r\gamma_0$, condition 1.10 (i) may be rewritten by using 1.10 (ii) as follows :

(2.9) $$\gamma_0 + d_r\gamma_1 = \text{" some linear combination of } g_i,\ i = 0, 1, \cdots, r \text{ "}$$

where

$$d_{r+1} = (5d_r - 1)/(d_r + 1), \quad r = 4, 5, \cdots$$
$$d_4 = 4.$$

By a simple calculation, d_r rapidly converges to $1/(2 - \sqrt{3})$ as $r \to \infty$.

Lemma 2. If $g(t) \in C^2[0, 1]$, then we have

(2.10) $$\|(I - P)g\| \leq Ch^2\|g''\| \text{ for sufficiently small } h.$$

Proof. Let $(Pg)(t) = \sum_{i=0}^{n} \gamma_i L_i(t)$, then

$$\begin{cases} \Delta^r\gamma_0 = 0 \\ (1/6)(\gamma_{i+1} + 4\gamma_i + \gamma_{i-1}) = g_i, \quad i = 1, 2, \cdots, n - 1 \\ \nabla^r\gamma_n = 0 \end{cases}$$

from which $\theta_i = \gamma_i - g_i$, $i = 0, 1, \cdots, n$ satisfy

$$\begin{cases} \Delta^r\theta_0 = -\Delta^r g_0 \\ (1/6)(\theta_{i+1} + 4\theta_i + \theta_{i-1}) = -(h^2/6)g''(\tau_i) \\ t_{i-1} \leq \tau_i \leq t_{i+1}, \quad i = 1, 2, \cdots, n - 1 \\ \nabla^r\theta_n = -\nabla^r g_n. \end{cases}$$

Since $\Delta^r g_0$, $\nabla^r g_n = O(h^2)\|g''\|$, in virtue of Lemma 1 we have

(2.11) $$(I - P)g(t) = (I - P_1)g(t) + O(h^2)\|g''\|$$

where $(P_1 g)(t) = \sum_{i=0}^{n} g_i L_i(t)$.

By a simple calculation, we have

(2.12) $$\|(I - P_1)g\| = O(h^2)\|g''\|$$

from which follows the desired result.

Since $r \geq 4$, similarly as in the proof of Lemma 2 we have

Lemma 3. If $g(t) \in C^4[0, 1]$, then we have

(2.13) $(Pg)(t_i) = g_i - (h^2/6)g_i'' + O(h^4)$, $i = 0, 1, \cdots, n$ for sufficiently small h.

As a consequence of this Lemma, we have

(2.14) $(Pg)(t) = (P_1 g)(t) - (h^2/6)(P_1 g'')(t) + O(h^4)$.

Now we turn to the investigation of asymptotic error estimation. Let $e_1(t) = \hat{x}(t) - \bar{x}_h(t)$ and $e_2(t) = \hat{w}(t) - \bar{w}_h(t)$, then in virtue of Theorem 1 we have

$$(2.15) \quad \begin{cases} e_1'(t) = e_2(t) \\ e_2'(t) = f_2(t)e_1(t) + f_3(t)e_2(t) \\ \qquad + (I - P)\hat{x}''(t) + R(t) \\ a_0 e_1(0) - b_0 e_2(0) = 0 \\ a_1 e_1(1) + b_1 e_2(1) = 0 \end{cases}$$

where $f_2(t) = f_x(t, \hat{x}(t), \hat{w}(t))$, $f_3(t) = f_w(t, \hat{x}(t), \hat{w}(t))$

$R(t) = (I - P)[\ f_2(t)e_1(t) + f_3(t)e_2(t)\ ]$.

Since $\|g'\| \leq C(\|g\| + \|g''\|)$ for any $g(t) \in C^2[0, 1]$, in virtue of Lemm 2 we have

(2.16) $\|R\| \leq Ch^2[\|e_1\| + \|e_1''\| + \|e_2\| + \|e_2''\|]$ for $h \leq h_0$

provided that h_0 is sufficiently small.

Since $\|e_1\|$, $\|e_2\| = O(h^2)$ and $e_1'' = e_2'$, we have

(2.17) $\|R\| \leq Ch^2[\ O(h^2) + \|e_2''\|]$ for $h \leq h_0$

where for $t_i \leq t \leq t_{i+1}$

$e_2''(t) = \hat{w}''(t) - \bar{w}_h''(t) = \hat{w}''(t) - (1/h)\{\ \bar{w}_h'(t_{i+1})$
$- \bar{w}_h'(t_i)\} = (1/h)\{\ e_2'(t_{i+1}) - e_2'(t_i)\} + O(h^2)$.

By (2.15), we have

(2.18) $|(1/h)\{\ e_2'(t_{i+1}) - e_2'(t_i)\ \}| \leq C[\ O(h^{-1})\|R\|$
$+ |(1/h)\{\ e_1(t_{i+1}) - e_1(t_i)\ \}| + |(1/h)\{\ e_2(t_{i+1})$
$- e_2(t_i)\ \}| + O(h^2)\ ]$.

By using again (2.15) we have

$$|(1/h)\{ e_2(t_{i+1}) - e_2(t_i) \}| \le (1/h)\int_{t_i}^{t_{i+1}} |e_2'(t)|dt$$

$$\le C[\|R\| + O(h^2)].$$

Therefore by (2.18) we have

$$(2.19) \quad |(1/h)\{ e_2'(t_{i+1}) - e_2'(t_i) \}| \le C[O(h^{-1})\|R\| + O(h^2)].$$

Combining (2.17) and (2.19) yields

$$(2.20) \quad \|R\| \le O(h^4) + O(h)\|R\| \qquad \text{for } h \le h_0$$

Finally we have the estimate of $\|R\|$ of the form

$$(2.21) \quad \|R\| = O(h^4) \qquad \text{for } h \le h_0$$

provided that if necessary, h_0 is replaced by a smaller constant.

Hence by (2.15) and (2.21) we have

$$(2.22) \quad \begin{bmatrix} e_1 \\ \\ e_2 \end{bmatrix} = \int_0^1 H(\cdot, s) \begin{bmatrix} 0 \\ \\ (I - P)\hat{x}''(s) \end{bmatrix} ds + \begin{bmatrix} O(h^4) \\ \\ O(h^4) \end{bmatrix}$$

where

$$(2.23) \qquad H(t, s) = \begin{cases} \Phi(t)[E - G^{-1}A_1\Phi(1)]\Phi^{-1}(s), & s \le t \\ \\ -\Phi(t)G^{-1}A_1\Phi(1)\Phi^{-1}(s), & s > t \end{cases}$$

(see the details of the Green function $H(t, s)$ in [5]).

In [6], we have obtained the following result:

$$(2.24) \quad \int_0^1 H_{i2}(t_j, s)(I - P_1)g(s) = -(h^2/12)\int_0^1 H_{i2}(t_j, s) \times g''(s)ds + O(h^4) \qquad \text{for } g \in C^2[0, 1]$$

$$i = 1, 2;\ j = 0, 1, \cdots, n - 1.$$

By using (2.22), (2.14) and (2.24), we have asymptotic expansions (1.17)-(1.18) at mesh points. Since cubic spline function $\bar{x}_h + (h^2/12)\phi$ satisfies

$$\bar{x}_h(t_j) + (h^2/12)\phi(t_j) = \hat{x}(t_j) + O(h^4)$$

$$\bar{x}'(t_j) + (h^2/12)\phi'(t_j) \ (= \bar{w}_h(t_j) + (h^2/12)\psi(t_j) \ )$$
$$= \hat{x}'(t_j) + O(h^4), \quad j = 0, 1, \cdots, n,$$

$\bar{x}_h + (h^2/12)\phi$ is considered to be a cubic spline interpolating to $\hat{x}$. Thus we have the desired asymptotic expansions (1.17)-(1.18) at any point $t \in [0, 1]$.

Now we consider the proof of Corollary of Theorem 2. From the definition of oerator P, we have

$$(2.27) \quad \begin{cases} \Delta^r \bar{x}_h''(t_0) = 0 \\ (1/6)\{ \bar{x}_h''(t_{i+1}) + 4\bar{x}_h''(t_i) + \bar{x}_h''(t_{i-1}) \} \\ = f(t_i, \bar{x}_h(t_i), \bar{w}_h(t_i)), \quad i = 1, 2, \cdots, n - 1 \\ \nabla^r \bar{x}_h''(t_n) = 0. \end{cases}$$

By (1.17)-(1.18), we have

$$(2.28) \qquad \begin{aligned} f(t_i, \bar{x}_h(t_i), \bar{w}_h(t_i)) &= f(t_i, \hat{x}(t_i) + (h^2/12)\phi(t_i), \\ \hat{w}(t_i) + (h^2/12)\psi(t_i)) &= \hat{x}''(t_i) - (h^2/12)\{ \\ f_2(t_i)\phi(t_i) + f_3(t_i)\psi(t_i) \} &+ O(h^4) \\ &= \hat{x}''(t_i) - (h^2/12)\{ \phi''(t_i) - \hat{x}^{(4)}(t_i) \} + O(h^4). \end{aligned}$$

Let $\zeta_i = \hat{x}''(t_i) - \bar{x}_h''(t_i)$, by (2.27)-(2.28) we have

$$(2.29) \begin{cases} \Delta^r \zeta_0 = O(h^r) \\ (1/6)(\zeta_{i+1} + 4\zeta_i + \zeta_{i-1}) = (h^2/12)\{ \phi''(t_i) + \hat{x}^{(4)}(t_i) \\ \} + O(h^4), \quad i = 1, 2, \cdots, n - 1 \\ \nabla^r \zeta_n = 0. \end{cases}$$

By means of Lemma 1, we have the desired asymptotic expansion (1.23).

3. Chopping Procedure

We have observed frequently that when problems are solved on a sequence of meshes, an acceptable solution is arrived at some regions before the problem as a whole has been solved. Our strategy is to chop off these regions where the solution is satisfactory and reformulate the new boundary value problems on the regions where the solution is still poor. Using asymptotic expansion (1.21) for the boundary conditions, we consider the new problems on the subintervals.

If $(h^2/12)|\bar{\phi}_h(t)| \leq \varepsilon$ (ε a desired tolerance)

for $t \in [0, a]$, $[b, 1]$ (a, b mesh points),

then let us chop off regions [0, a] and [b, 1] and consider the following problem :

(3.1) $x''(t) = f(t, x(t), x'(t)), \quad a \leq t \leq b$

(3.2) $x(a) = \bar{x}_h(a) + (h^2/12)\bar{\phi}_h(a)$

(3.3) $x(b) = \bar{x}_h(b) + (h^2/12)\bar{\phi}_h(b)$

where possibly we have by (1.21)-(1.22)

(3.4) $|\hat{x}(a) - \{ \bar{x}_h(a) + (h^2/12)\bar{\phi}_h(a) \}| \ll \varepsilon$

(3.5) $|\hat{x}(b) - \{ \bar{x}_h(b) + (h^2/12)\bar{\phi}_h(b) \}| \ll \varepsilon.$

If $(h^2/12)|\bar{\phi}_h(t)| \leq \varepsilon$ for $t \in [a, b]$ (a, b mesh points), then we consider the following new problems on the remaining regions [0, a] and [b, 1] :

(3.6) $x''(t) = f(t, x(t), x'(t)), \quad 0 \leq t \leq a$

(3.7) $a_0x(0) - b_0x'(0) = c_0$

(3.8) $x(a) = \bar{x}_h(a) + (h^2/12)\bar{\phi}_h(a).$

(3.9) $x''(t) = f(t, x(t), x'(t)), \quad b \leq t \leq 1$

(3.10) $x(b) = \bar{x}_h(b) + (h^2/12)\bar{\phi}_h(b)$

(3.11) $a_1x(1) + b_1x'(1) = c_1.$

By reducing h to h/2 and using P, we consider the numerical solution of the problems (3.1)-(3.3), or (3.6) -(3.8) and (3.9)-(3.11). The successive use of this procedure will give the approximate solution $\bar{x}_h(t)$ so that

$$(3.12) \qquad \|\hat{x} - \{ \bar{x}_h + (h^2/12)\bar{\phi}_h\}\| << \|\hat{x} - \bar{x}_h\| \leq \varepsilon.$$

4. Numerical Illustration

In this section we shall consider the application of the above asymptotic expansions (1.30)-(1.31) to a posteriori improvement of spline approximations of solutions of two point boundary value problems. And further we demonstrate the chopping procedure with some numerical results. These results conform the theoretical accuracies established in previous sections. The rate of decrease of the error $O(h^\alpha)$, where α is computed from the results from $h = 1/16$ to $1/32$, is given in parentheses in each Table.

As our examples, we choose

Problem 1.

$$x'' = \tfrac{1}{2}(x^2 + x'^2)/e^t$$

$$x(0) - x'(0) = 0$$

$$x(1) + x'(1) = 2e.$$

Problem 2.

The same differential equation as in Problem 1 subject to the boundary conditions:

$$x(0) = 1, \quad x(1) = e.$$

The exact solutions of the above problems are e^t.

In the following Tables, the left and right hand sides of $(\cdots) \to (\cdots)$ mean $\max_i |\hat{x}^{(k)}(t_i) - \bar{x}_h^{(k)}(t_i)|$ and $\max_i |\hat{x}^{(k)}(t_i) - \{\bar{x}_h(t_i) + (h^2/12)\bar{\phi}_h(t_i)\}|$, $k = 0, 1$, respectively.

Table 4.1

The observed maximum errors for function values.

h	1/16	1/32	α
Problem 1	0.209-3* → 0.343-6	0.522-4 → 0.213-7	2.0 → 4.0
Problem 2	0.612-4 → 0.988-7	0.153-4 → 0.619-8	2.0 → 4.0

* we write 0.209×10^{-3} by 0.209-3.

Table 4.2

The observed maximum derivative errors.

h	1/16	1/32	α
Problem 1	0.158-3 → 0.341-6	0.464-4 → 0.210-7	2.0 → 4.0
Problem 2	0.344-3 → 0.865-6	0.861-4 → 0.529-7	2.0 → 4.0

The above stated method is also applicable to the numerical solution of the nonlinear boundary value problem having the singularity at the origin :

$$x''(t) + (\kappa/t)x'(t) + f(t, x(t)) = 0, \quad 0 < t \le 1$$

$$x'(0) = 0 \quad \text{and} \quad x(1) = c_1$$

with $\kappa = 0$, 1, or 2, respectively.

While Theorems 1 - 3 are not assured for the above problem, no numerical difficulties are encountered.

Problem 3. We treat the nonlinear problem :

$$x'' + (2/t)x' + x^5 = 0 \ , \ c_1 = \sqrt{3}/2.$$

The unique solution is

$$\hat{x} = 1/\sqrt{1 + t^2/3} \ .$$

Problem 4. Consider another nonlinear problem :

$$x'' + (1/t)x' + \exp(x) = 0, \quad 0 < t \leq 1$$

$$c_1 = 0.$$

The solution are $\hat{x} = 2 \ln[(B + 1)/(Bt^2 + 1)]$, where $B = 3 \pm 2\sqrt{3}$. In the following Table, we only list up numerical results for the smaller solution.

Table 4.3

The observed maximum errors for function values.

h	1/16	1/32	α
Problem 3	0.220-4 → 0.279-6	0.546-5 → 0.175-7	2.0 → 4.0
Problem 4	0.491-4 → 0.169-7	0.123-4 → 0.105-8	2.0 → 4.0

Table 4.4

The observed maximum derivative errors.

h	1/16	1/32	α
Problem 3	0.368-4 → 0.372-6	0.962-5 → 0.229-7	2.0 → 4.0
Problem 4	0.668-4 → 0.803-7	0.169-4 → 0.498-8	2.0 → 4.0

Now we consider the application of chopping procedure to the following problems in which we take a desired tolerance $\varepsilon = 10^{-4}$ and $h = 1/32$ as starting mesh sizes.

Problem 5. First we consider the singular perturbation problem:

$$10^{-4}x'' - x = 1, \quad 0 \leq t \leq 1$$

$$x(0) = x(1) = 1.$$

There exists a unique solution symmetric about $t = 1/2$ and having boundary layers of thickness 0.01 at $t = 0$ and $t = 1$.

Problem 6.

$$10^{-4}x'' + (1 - \tfrac{1}{2}t)x' - \tfrac{1}{2}x = 0, \quad 0 \leq t \leq 1$$

$x(0) = 0$ and $x(1) = 1$.

The exact solution is approximately $1/(2 - t)$ on $(0, 1]$ and has a boundary layer of thickness 10^{-4} at $t = 0$.

Problem 7. Now we consider a real problem :

$$x'' + \{ 3\cot an(t) + 2\tan(t) \}x' + 0.7x = 0$$

$$30^\circ \leq t \leq 60^\circ$$

subject to the boundary conditions

$$x(30^\circ) = 0 \quad \text{and} \quad x(60^\circ) = 5.$$

The solution curve has a sharp spike approximately at 30.66° with the magnitude of the solution at this point $283.26\cdots$.

In the following Table $a = 30^\circ$ and N is the maximum number of grid points of the remaining subintervals, that is, we have to solve a linear system of order $N + 3$ at least one time. In Problem 5, only the half of the interval of $[0, 1]$ is covered because of the symmetry.

Table 4.5

Remaining subintervals.

	Problem 5	Problem 6	Problem 7
	$[\ 0\ ,\ \frac{32}{32}\ ]$	$[\ 0,\ \frac{32}{32}\ ]$	$[\ a,\ (1+\frac{32}{32})a\ ]$
	$[\ 0\ ,\ \frac{6}{64}\ ]$	$[\ 0,\ \frac{64}{64}\ ]$	$[\ a,\ (1+\frac{53}{64})a\ ]$
	$[\ 0\ ,\ \frac{10}{128}\ ]$	$[\ 0,\ \frac{128}{128}\ ]$	$[\ a,\ (1+\frac{71}{128})a\ ]$
	$[\ 0\ ,\ \frac{16}{256}\ ]$	$[\ 0,\ \frac{242}{256}\ ]$	$[\ a,\ (1+\frac{21}{256})a\ ]$
	$[\ 0\ ,\ \frac{22}{512}\ ]$	$[\ 0,\ \frac{150}{512}\ ]$	$[\ a,\ (1+\frac{37}{512})a\ ]$
	$[\ \frac{3}{1024},\ \frac{22}{1024}\ ]$	$[\ 0,\ \frac{82}{1024}\ ]$	$[\ a,\ (1+\frac{64}{1024})a\ ]$
		$[\ 0,\ \frac{49}{2048}\ ]$	$[\ a,\ (1+\frac{109}{2048})a\ ]$
		$[\ 0,\ \frac{19}{4096}\ ]$	$[\ a,\ (1+\frac{182}{4096})a\ ]$
		$[\ 0,\ \frac{8}{8192}\ ]$	$[\ a,\ (1+\frac{290}{8192})a\ ]$
		$[\ 0,\ \frac{12}{16384}\ ]$	$[\ (1+\frac{2}{16384})a,\ (1+\frac{397}{16384})a\ ]$
		$[\ 0,\ \frac{17}{32768}\ ]$	
		$[\ 0,\ \frac{21}{65563}\ ]$	$[\ (1+\frac{29}{32768})a,\ (1+\frac{470}{32768})a\ ]$
N	44	484	882

From numerical results in this paper and [6], it is certain that collocation method and difference methods are almost the same in effectiveness. Since the coefficient matrix associatd with the former is of band-width three and the coefficient one associated with the latter is of band-width five, collocation method seems to be more economical than difference method.

References

1. J. Ahlberg, E. Nilson and J. Walsh: The theory of splines and their applications. New York, Academic Press (1967).
2. A. Aziz (ed.): Numerical solutions of boundary value problems for ordinary differential equations. New York, Academic Press (1974).
3. D. Jacobs (ed.): The state of the art in numerical analysis. New York, Academic Press (1977).
4. R. Russell and J. Christiansen: Adaptive mesh selection strategies for solving boundary value problems. SIAM J. Numer. Anal. 15, 59-80 (1978).
5. M. Sakai: Piecewise cubic interpolation and two-point boundary value problems. Publ. RIMS. Kyoto Univ. 7, 345-362 (1972).
6. ———: A posteriori Improvement of cubic spline approximate solution of two-point boundary value problem. Collocation method. (to appear in Publ. RIMS. Kyoto Univ.).

On the numerical integration of oscillatory functions

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1. Introduction

On the numerical integration of oscillatory functions , Euler's transformation is often used in the case of slow convergence. We have computed some examples of the numerical integration by the use of Euler's transformation :

$$\sum_{n=0}^{\infty} (-1)^n An = \sum_{n=0}^{\infty} [(-1)^n \Delta^n Ao / 2^{n+1}]$$

where $An \downarrow 0$, Δ is forward difference operator.

$$An = \left| \int_{x_n}^{x_{n+1}} f(x)\, dx \right| \qquad (n = 0,1,2,\cdots) \text{ ------ } (1)$$

f(x) : Oscillatory function

x_n : n-th zero point of f(x)

so, we have found an elementary proof of $\int_0^{\infty} (\sin x)/x \, dx = \pi/2$, and the sequence of functions which can be computed the exact value by the composite trapezoidal rule.

2. The peculiarity of the integral of (sin x)/x on [0 , ∞)

On the numerical caluculation of $I = \int_0^{\infty} (\sin x)/x \, dx$, it is necessary to compute several terms ' An ' acçulately for computing the value I exactly in general, where

$$An = \left| \int_{n\pi}^{(n+1)\pi} (\sin x)/x \, dx \right| \qquad (n = 0,1,2, \cdots) \text{ -------- } (2)$$

But the example of the function '(sin x)/x',it is known that the computed value S(N) by the trapezoidal rule of division N of the interval $[n\pi, n\pi+\pi]$ is equal to the exact value 'I',([3],[4],[5]) where ,

$$S(N) = \pi/2N + (\pi/N)\sum_{K=1}^{\infty}(\sin k\pi/N)/(k\pi/N)$$

F.Stenger[4] showed that the convergence of infinite sum S(N) is so slow that the approximation(trapezoidal rule) of this integral is practically useless.But by the use of the Euler's transformation, we could compute the exact value of the infinite sum down to 15 decimals with 40 terms.That is to say,in computing the values 'An' by using the composite trapezoidal rule of division N on the sub-interval $[n\pi,(n+1)\pi]$, we could compute the same exact value $\pi/2$ of the integration I in spite of many different values of An , which depends on the number of division N of the interval $[n\pi,(n+1)\pi]$ .Even in the simplest case N=2,we can calculate $\pi/2$ down to 15 decimals .

Table 1. The comparison by the division N

N	2	3	4
A 0	0.1785398163397448D+01	0.1822636881274957D+01	0.1835508123280788D+01
A 1	0.3333333333333333D+00	0.3897114317029974D+00	0.4091032773591973D+00
A 2	0.2000000000000000D+00	0.2319710902994032D+00	0.2428498547851611D+00
A 3	0.1428571428571429D+00	0.1653321225406656D+00	0.1729618528297167D+00
A 4	0.1111111111111111D+00	0.1284762961658233D+00	0.1343662184741800D+00
A 5	0.9090909090909090D-01	0.1050692585473767D+00	0.1098700700346864D+00
⋮	⋮	⋮	⋮
A35	0.1408450704225352D-01	0.1626374634156986D-01	0.1700199789730984D-01
A36	0.1369863013698630D-01	0.1581814540690508D-01	0.1653616377158824D-01
A37	0.1333333333333333D-01	0.1539631130305773D-01	0.1609517612735222D-01
A38	0.1298701298701299D-01	0.1499639192460309D-01	0.1567709867589977D-01
A39	0.1265822784810126D-01	0.1461672273870616D-01	0.1528019123266534D-01
DIRECT	1.5645473025959065D+00	1.5635804500100140D+00	1.5632529148298061D+00
EULER	1.5707963267948965D+00	1.5707963267948963D+00	1.5707963267948964D+00
TRUE	1.5707963267948966D+00	1.5707963267948966D+00	1.5707963267948966D+00

DIRECT : Direct sum of $(-1)^n$ An (n = 0,1,2,···,39)
EULER : Value of the integration by Euler's transformation
TRUE : The true value of the integration

3. The proof of $\int_0^{\infty}(\sin x)/x\,dx = \pi/2$

By the composite trapezoidal rule of division N on each sub-intervals of the length π, we calculate the value as the following .

$$S(N) = \pi/2N + (\pi/N)\sum_{k=1}^{\infty}(\sin k\pi/N)/(k\pi/N) \text{ ---------- } (3)$$

and we have the following expression .

$$I = \lim_{N\to\infty} S(N) \text{ ----------- } (4)$$

Now,we'll prove ,'the value of S(N) does not depend on the division N and is always equal to the constant $\pi/2$ ' , by the formula of trigonometric series ,

$$\sum_{k=1}^{\infty} (\sin kx)/k = (\pi - x)/2, \quad \text{for } 0 < x < 2\pi \text{ ------- } (5)$$

So it is easy to prove ' $S(N) = \pi/2$,for any integer $N \geq 2$' .

$$I = \lim_{N\to\infty} S(N) = \pi/2$$

□

4. About the proof of the $\int_0^{\infty}(\sin x)/x\,dx = \pi/2$

Several proof of ' $\int_0^{\infty}(\sin x)/x\,dx = \pi/2$' were discussed by G.H.Hardy [2].

The marks of the proof by G.H.Hardy depend mainly on the exchange of two limits,—the double limit problems . But in our proof,we use the formula (5). Abel's continuity theorem on the complex power series may be used to prove the formula (5).

Moreover, substituting '$x = \pi/N$' in the formula (5), we got the expression below .

$$\sum_{k=1}^{\infty} (\sin k(\pi/N))/k = (\pi - \pi/N)/2 \text{ ---------- } (7)$$

And this formla (7) is quite the same as the calculation of the integration $I = \int_0^{\infty}(\sin x)/x\,dx$ by the composite trapezoidal rule of division N on the sub-interval of the length π .

5. Expansion of our proof

Now we will compare two forms below,

$I = \int_0^{\infty} (\sin x)/x \, dx$ and trigonometric series ,

$$\sum_{n=1}^{\infty} (\sin nx)/n = (\pi - x)/2,$$

($0 < x < 2\pi$) which corresponds to I .

We will write $In = \int_0^{\infty} Fn(x) \, dx$ by n-times integrating I by parts,

and $Fn = (L^{n-1} \sin x)/ x^n$ $(n = 1, 2, \cdots)$,where

L : integration operator

i.e. $L f(x) = \int_0^{x} f(x) \, dx, \quad L^0 f(x) = f(x)$

Sn is defined as a calculated value of In by composite trapezoidal rule of step size 'h' $(0 < h < \pi)$.

Then we find a result as the following.

' Sn does not depend on h and is equal to In exactly '

[proof]

(i) We discuss the trigonometric series (1) which corresponds to $I = \int_0^{\infty} (\sin x)/x \, dx$.

$$\sum_{j=1}^{\infty} (\sin jx)/j = (\pi - x)/2 \quad (0 < x < 2\pi) \text{ ------- } (8)$$

The left side member of the formula (8) converges uniformly, so we can repeat termwise integration (n-1) times on (8)

$$L^{n-1} \left(\sum_{j=1}^{\infty} (\sin jt)/j \right) = \sum_{j=1}^{\infty} \left[L^{n-1} \sin x / j^n \right]_{x=jt} = L^{n-1} [(\pi - x)/2]$$

So we find the formula (9),

$$\sum_{j=1}^{\infty} \left[L^{n-1} \sin x / j^n \right]_{x=jt} = \pi \cdot t^{n-1} / 2(n-1)! - t^n / 2 \cdot n! \text{ ------ } (9)$$

Remark: The formula(9) is ,what we call,equivallent formula of In.

(ii)

$$Sn = h/2\ Fn(0) + h \sum_{j=1}^{\infty} F(jh)$$

$$= h/2\ Fn(0) + h^{1-n} \sum_{j=1}^{\infty} [L^{n-1} \sin x]/j^{n} \Big|_{x=jh} \qquad \text{----------} \ (10)$$

and

$$Fn(0) = 1 / n! \qquad \text{---------------} \ (11)$$

from (9),(10),(11), we find (12).

$$Sn = h/2 \cdot n! + h^{1-n} [\pi \cdot h^{n-1} / 2\ (n-1)! - h/2\ n!]$$

$$= \pi / 2\ (n-1)! \qquad \text{---------------} \ (12)$$

(iii) Expression (9) is regarded as a function of variable h,

$Sn = Sn(h)$.

$$In = \lim_{h \to 0} Sn(h) = \pi / 2 \cdot (n-1)!$$

Hence, Sn is independent of h and equal to In.

$$In = Sn = \pi / 2 \cdot (n-1)! \quad (n = 1,2,3,\cdots)$$

□

Table 2. Example of Functions

n	Fn(x)	Series
1	$(\sin x)/x$	$\sum_{k=1}^{\infty} (\sin kx)/k = (\pi - x)/2$
2	$(1-\cos x)/x^2$	$\sum_{k=1}^{\infty} (1-\cos kx)/k^2 = \frac{\pi}{2} x - \frac{1}{4} x^2$
3	$(x-\sin x)/x^3$	$\sum_{k=1}^{\infty} (kx-\sin kx)/k^3 = \frac{\pi}{4} x^2 - \frac{1}{12} x^3$
4	$(\frac{1}{2} x^2 - 1 + \cos x)/x^4$	$\sum_{k=1}^{\infty} (\frac{1}{2} k^2 x^2 - 1 + \cos kx)/k^4 = \frac{\pi}{12} x^3 - \frac{1}{48} x^4$
⋮	⋮	⋮

But to our regret ,if the number'n' is not less than 3,the function Fn(x) is no longer oscillatory ,however,'In' is equal to 'Sn'exactly. And then,the method of the Euler's transformation can't be applicable to non-alternating series. It is true the property that the computed value by the trapezoidal rule does not depend on step size 'h' and is equal to the exact value,is interesting,but those examples need so much computational quantity that the trapezoidal rule is practically useless.

6. Conclusion

So far as calculating the integration, $I = \int_0^{\infty} (\sin x)/x \, dx$ by the composite trapezoidal rule of division N on sub-intervals , there is a peculiarity that we could compute $\pi/2$ for any positive integer N (N>1). So we have come to the conclusion that the function'(sin x)/x' is inappropriate for the test function of formulae of numerical integration, and we must pay attention to this point .

Secondly,the sequence of functions which are generated by n-times integrating'(sin x)/x 'by parts,have the property that the computed values by the trapezoidal rule don't depend on step size h and are equal to their exact values . But 'n' is not less than 3, these functions are no longer oscillatory. And then, we can't make use of Euler's transformation. So,it is difficult to satisfy the prescribed precision by the trapeziodal rule.And we should compute these values by the other formulae.

[REFERENCES]

[1] Davis,P.J.,/Rabinowitz,P.,/Mori,M.(translator)

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NIPPON COMPUTER KYOKAI (1981)

[2] Hardy,G.H.

THE INTEGRAL $\int_0^\infty$ (sin x)/x dx.

THE MATHEMATICAL GAZETTE,p.98-p.103 (1909)

[3] Boas,R.P,/Pollard,H,

Continuas Analogues of Series.

The American Mathematical Monthly, Vol.80 (1973) pp.18-25

[4] Stenger,F.,

Integration Formulae Based on the Trapezoidal Formula

J.Inst.Maths.Applics Vol.12 (1973) pp.103-114

[5] Henrici,P.,

Applied and Computational Complex Analysis, Vol.2

John Wiley & Sons, pp.272-276 (1977)

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Certain classification for non-isomorphic solutions of 2-design

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Abstract

We introduce a new mapping (H^s-transformation) on a 2-design. Theorem 1 plays a fundamental role for this investigation in which we attempted to classify 2-designs having the same parameters. In order to decide whether 2-designs are isomorhpic, we developed a criterion (Corollary 1) by the use of H^s-transformations. When the Steiner system has H^s-transformations it shows s=1 and the design is a Steiner triple system (Theorem 3). If the design is symmetrical, its H^s-transformation will have remarkable features (cf. Theorems 4 and 5).

In general, it is difficult to classify 2-designs by H^s-transformations, however, we produced six examples (two of which are symmetrical) which will decide a class including one design. Some of the designs have not been fully reported previously.

1. Characteristics of H^s-transformations

Theorem 1. If we let points of a 2-design D be 1,2,...,v, and blocks be $B_1, B_2, \ldots, B_b$, and if we assume as below

$$B_1(a_1,\ldots,a_p,\ b_1^1,\ldots,b_m^1,\ c_1^1,\ldots,c_n^1,\ d_1^1,\ldots,d_s^1)$$
$$B_2(a_1,\ldots,a_p,\ b_1^2,\ldots,b_m^2,\ c_1^2,\ldots,c_n^2,\ d_1^2,\ldots,d_s^2)$$
$$B_3(a_1,\ldots,a_p,\ b_1^3,\ldots,b_m^3,\ c_1^3,\ldots,c_n^3,\ d_1^3,\ldots,d_s^3)$$
$$B_4(a_1,\ldots,a_p,\ b_1^4,\ldots,b_m^4,\ c_1^4,\ldots,c_n^4,\ d_1^4,\ldots,d_s^4)$$

where $p \geqq 0$, $m \geqq 1$, $n \geqq 1$, $s \geqq 1$, $p+m+n+s=k$ and $b_i^1=b_i^2$, $b_i^3=b_i^4$ $(i=1,2,\ldots,m)$, $c_j^1=c_j^3$, $c_j^2=c_j^4$ $(j=1,2,\ldots,n)$, $d_l^1=d_l^4$, $d_l^2=d_l^3$ $(l=1,2,\ldots,s)$, (the intersections of the four blocks are only $a_1,a_2,\ldots,a_p$). Then for B_1,B_2,B_3,B_4 if we try to substitute B_1',B_2',B_3',B_4' as below

$$B_1'(a_1,\ldots,a_p,\ b_1^1,\ldots,b_m^1,\ c_1^1,\ldots,c_n^1,\ d_1^2,\ldots,d_s^2)$$
$$B_2'(a_1,\ldots,a_p,\ b_1^2,\ldots,b_m^2,\ c_1^2,\ldots,c_n^2,\ d_1^1,\ldots,d_s^1)$$
$$B_3'(a_1,\ldots,a_p,\ b_1^3,\ldots,b_m^3,\ c_1^3,\ldots,c_n^3,\ d_1^4,\ldots,d_s^4)$$
$$B_4'(a_1,\ldots,a_p,\ b_1^4,\ldots,b_m^4,\ c_1^4,\ldots,c_n^4,\ d_1^3,\ldots,d_s^3)$$

This new incidence structure D' will turn out also to be a 2-design.

Proof. It is sufficient that the inner product of two row vectors of M(D') (M(D') is an incidence matrix of D') is λ. If the cardinality of row vectors which include d_i^1th-row and d_i^2th-row $(i=1,2,\ldots,s)$ is 0,1,2, then the inner product is λ respectively.

We shall denote this transformation as below (up to isomorphism).

$1\ 2\ 3\ 4 \mid a_1\ldots d_s^2,\ a_1\ldots d_s^1,\ a_1\ldots d_s^4,\ a_1\ldots d_s^3 \mid D'$

or more briefly

$1\ 2\ 3\ 4 \mid D'$

We call the mapping of Theorem 1 an H^s-transformation. Furthermore, if it is $D \ncong D'$ we can call it a proper H^s-transformation, and if it is $D \cong D'$ we can call it an identity H^s-transformation. We shall represent image D' of D by H^s-transformation α^s as $D\alpha^s$. If s=1, H^s, α^s, $\overset{s}{\sim}$, $[D]^s$ will only show H, α, $\sim$, [D]. If D has H^s-transformations and the cardinalities of images $D_1, D_2, \ldots, D_l$ by these transformations are $n_1, n_2, \ldots, n_l$ respectively (up to isomorphism) we will denote as below.

$$D \overset{s}{\sim} n_1 D_1 + n_2 D_2 + \ldots + n_l D_l$$

From the definition of H^s-transformation we get immediately Corollary 1,2,3.

Corollary 1. Suppose we have $D \cong D'$, then the images of D by H^s-transformation will coincide with those of D' by H^s-transformation (up to isomorphism). In particular the cardinality of the images of D is equal to that of images of D'.

Corollary 2. If there exists an H^s-transformation from blocks B_i of D to B_i' of D', then a mapping from B_i' of D' to B_i of D is also an H^s-transformation.

Corollary 3. If a design D has an H^s-transformation α^s, then its comlementary design D^c has a corresponding H^s-transformation (this transformation and its image are also denoted by α^s and $D^c\alpha^s$), and $(D\alpha^s)^c \cong D^c\alpha^s$ holds. If α^s is a proper H^s-transformation, then α^s of D^c is also proper.

Using Corollary 3, we have Theorem 2.

Theorem 2. If $D \overset{s}{\sim} n_1 D_1 + n_2 D_2 + \ldots + n_l D_l$, then

$$D^c \overset{s}{\sim} n_1 D_1^c + n_2 D_2^c + \ldots + n_l D_l^c$$

Note that Theorem 2 is particularly useful in the case of v=2k, and that it serves to self-complementary D as well. An H^s-transformation of Theorem 1 means H^m-transformation or an H^n-transformation. These transformations are called conjugates of each other. If an H^m-transformation or H^n-transformation is a conjugate to an H^s-transformation α^s it is represented α^m or α^n.

From assumption of Theorem 1, $1 \leqq s \leqq k-2$ is clearly found and the next follows.

Corollary 4. If D has an H^s-transformation α^s (s=k-2 or k-3), then D has a conjugate H-transformation α.

When $\lambda=1$, we have following.

Theorem 3. If a Steiner system S(2,v,k) has H^s-transformations, then s=1 and k=3.

Proof. If S has an H^s-transformation of Theorem 1, the cardinality of the intersection of two blocks is 0 or 1 as $\lambda=1$. Therefore p=0, m=n=s=1 and k=3.

Example 1. Let v=6, k=3, b=10, r=5, $\lambda=2$ and
D: B_1(123), B_2(126), B_3(134), B_4(145), B_5(156), B_6(235), B_7(245), B_8(246), B_9(346), B_{10}(356)

H-transformation of D

148·10	124	135	236	456	D
1579	125	136	234	456	D
237·10	124	136	256	345	D
2469	125	146	236	345	D
3568	135	146	234	256	D

therefore $D \sim 5D$

2. H^s-transformation of symmetrical designs

Lemma. If a design $D(v,k,\lambda)$ with $k \leqq \frac{v}{2}$ is symmetrical, then $v \neq 2k$ and $k \geqq 2\lambda+1$.

Proof. Since D is symmetrical, it follows $k(k-1)=\lambda(v-1)$. Suppose $v=2k$ then $\lambda=\frac{k(k-1)}{2k-1}$ is not an integer. We can assume $v \geqq 2k+1$, and so $2\lambda+1= \frac{2k(k-1)}{v-1} + 1 \leqq k$

Theorem 4. If a symmetrical design $D(v,k,\lambda)$ has H^s-transformations of Theorem 1, then $m=n=s$ and $2s=k-\lambda$

Proof. Any inner product of column vectors of M(D) is also λ. By counting in the inner product of the 1st column vector and the 2nd one , and that of the 1st column vector and the 3rd one, and that of the 1st column vector and the 4th one respectively, we obtain $\lambda=p+m=p+n=p+s$, therefore $m=n=s$ and $k-\lambda=2s$.

Theorem 5.

(1) For any given value of s, there exist finitely many symmetrical designs with $k < \frac{v}{2}$ having H^s-transformations.

(2) If a symmetrical design $D(v,k,\lambda)$ with $k < \frac{v}{2}$ has H^s-transformations, the following table follows:

s	v	k	λ	s	v	k	λ
1	7	3	1	6	47	23	11
2	16	6	2	7	55	27	13
2	15	7	3	8	66	26	10
3	25	9	3	8	64	28	12
3	23	11	5	8	70	24	8
4	31	15	7	8	63	31	15
5	41	16	6	9	71	35	17
5	39	19	9	.	.	.	.

Proof.

(1) From Lemma and Theorem 4 make $2\lambda+1 \leqq \lambda+2s$ so

$s \leqq \lambda \leqq 2s-1$ ... 1)

Then the number of λ with any given value of s is finite and so there are finitely many symmetrical designs with H^s-transformations.

(2) Let s=1 then $\lambda=1$ from 1), hence k=3, v=7. Let s=2 then k=6, v=16 in case $\lambda=2$; or k=7, v=15 in case $\lambda=3$, and so on. Thus the table required follows.

Example 2. If a symmetrical design $D(v,k,\lambda)$ has H-transformations, then from Theorem 5(2), v=7, k=3, $\lambda=1$. A design D(7,3,1) has three identical H-transformations, thus a symmetrical design doesn't have proper H-transformations.

3. Classification of 2-designs

Let D_1 be an image of D by an H^s-transformation, D_2 be an image of D_1 by an H^s-transformation, ..., D_l be an image of D_{l-1} by an H^s-transformation, then we can define that D_l stands in the same class as D. Thus it is possible to classify non-isomorphic 2-designs having the same parameters. This class can be called H^s class and we describe $[D]^s$ the class which D belong to.

Example 3. Let v=8, k=4, b=14, r=7, λ=3 and

D^0: B_1(1234), B_2(1256), B_3(1278), B_4(1357), B_5(1368), B_6(1458), B_7(1467), B_8(2358), B_9(2367), B_{10}(2457), B_{11}(2468), B_{12}(3456), B_{13}(3478), B_{14}(5678)

D^4: B_1(1234), B_2(1257), B_3(1268), B_4(1356), B_5(1378), B_6(1458), B_7(1467), B_8(2358), B_9(2367), B_{10}(2457), B_{11}(2468), B_{12}(3456), B_{13}(3478), B_{14}(5678)

D^6: B_1(1234), B_2(1258), B_3(1267), B_4(1356), B_5(1378), B_6(1458), B_7(1467), B_8(2357), B_9(2368), B_{10}(2457), B_{11}(2468), B_{12}(3456), B_{13}(3478), B_{14}(5678)

D^7: B_1(1234), B_2(1238), B_3(1267), B_4(1356), B_5(1458), B_6(1467), B_7(1578), B_8(2357), B_9(2457), B_{10}(2468), B_{11}(2568), B_{12}(3456), B_{13}(3478), B_{14}(3678)

These designs are non-isomorphic of each other, because the cardinalities of the inner product are 0,4,6,7, respectively, in case the inner product of column vectors at the incidence matrix is three. Furthermore these designs are all self-complementary. Thus the next table follows:

H-transformations of D^0

1247	1235	1246	1347	1567	D^4
128·11	1235	1246	2348	2568	D^4
1256	1236	1245	1348	1568	D^4
129·10	1236	1245	2347	2567	D^4
1346	1237	1248	1345	1578	D^4
130·11	1237	1248	2346	2678	D^4
1357	1238	1247	1346	1678	D^4
138·10	1238	1247	2345	2578	D^4
148·13	1235	1347	2348	3578	D^4
149·12	1237	1345	2346	3567	D^4
159·13	1236	1348	2347	3678	D^4
158·12	1238	1346	2345	3568	D^4
16·10·13	1245	1348	2347	4578	D^4
16·11·12	1248	1345	2346	4568	D^4
17·11·13	1246	1347	2348	4678	D^4
17·10·12	1247	1346	2345	4567	D^4
2345	1257	1268	1356	1378	D^4
23·10·11	1257	1268	2456	2478	D^4
2367	1258	1267	1456	1478	D^4
2389	1258	1267	2356	2378	D^4
24·10·12	1257	1356	2456	3457	D^4
248·14	1235	1567	2568	3578	D^4
25·11·12	1268	1356	2456	3468	D^4
259·14	1236	1568	2567	3678	D^4
268·12	1258	1456	2356	3458	D^4
26·10·14	1245	1568	2567	4578	D^4
279·12	1267	1456	2356	3467	D^4
27·11·14	1246	1567	2568	4678	D^4
34·10·13	1257	1378	2478	3457	D^4
349·14	1237	1578	2678	3567	D^4
35·11·13	1268	1378	2478	3468	D^4
358·14	1238	1678	2578	3568	D^4
368·13	1258	1478	2378	3458	D^4
36·11·14	1248	1578	2678	4568	D^4
379·13	1267	1478	2378	3467	D^4
37·10·14	1247	1678	2578	4567	D^4
4567	1358	1367	1457	1468	D^4
4589	1358	1367	2357	2368	D^4
45·12·13	1356	1378	3457	3468	D^4
468·10	1358	1457	2357	2458	D^4
46·12·14	1345	1578	3567	4568	D^4
479·10	1367	1457	2357	2467	D^4
47·13·14	1347	1567	3578	4678	D^4
568·11	1358	1468	2368	2458	D^4
56·13·14	1348	1568	3678	4578	D^4
579·11	1367	1468	2368	2467	D^4
57·12·14	1346	1678	3568	4567	D^4
67·10·11	1457	1468	2458	2467	D^4
67·12·13	1456	1478	3458	3467	D^4
89·10·11	2357	2368	2458	2467	D^4
89·12·13	2356	2378	3458	3467	D^4
8·10·12·14	2345	2578	3568	4567	D^4
8·11·13·14	2348	2568	3578	4678	D^4
9·10·13·14	2347	2567	3678	4578	D^4
9·11·12·14	2346	2678	3567	4568	D^4
10·11·12·13	2456	2478	3457	3468	D^4

H-transformations of D^4

1247	1235	1247	1345	1567	D^6
1256	1237	1245	1348	1578	D^6
1346	1236	1248	1345	1568	D^6
1357	1238	1246	1347	1678	D^6
16·10·13	1245	1348	2347	4578	D^6
16·11·12	1248	1345	2346	4568	D^6
17·10·12	1247	1346	2345	4567	D^6
17·11·13	1246	1347	2348	4678	D^6
2345	1256	1278	1357	1368	D^0
23·12·13	1256	1278	3457	3468	D^4
2367	1258	1267	1457	1468	D^6
2389	1258	1267	2357	2368	D^6
24·11·13	1267	1357	2478	3468	D^4
248·14	1235	1567	2578	3568	D^6
25·11·12	1278	1357	2456	3468	D^4
259·14	1237	1578	2567	3678	D^6
34·10·13	1256	1368	2478	3457	D^4
349·14	1236	1568	2678	3567	D^6
35·10·12	1278	1368	2456	3457	D^4
358·14	1238	1678	2568	3578	D^6
45·10·11	1357	1368	2456	2478	D^4
4567	1358	1367	1456	1478	D^6
4589	1358	1367	2356	2378	D^6
67·10·11	1457	1468	2458	2467	D^6
67·12·13	1456	1478	3458	3467	D^6
89·10·11	2357	2368	2458	2467	D^6
89·12·13	2356	2378	3458	3467	D^6
8·10·12·14	2345	2578	3568	4567	D^6
8·11·13·14	2348	2568	3578	4678	D^6
9·10·13·14	2347	2567	3678	4578	D^6
9·11·12·14	2346	2678	3567	4568	D^6
10·11·12·13	2456	2478	3457	3468	D^0

H-transformations of D^6

16·10·13	1245	1348	2347	4578	D^7
16·11·12	1248	1345	2346	4568	D^7
17·11·13	1246	1347	2348	4678	D^7
17·10·12	1247	1346	2345	4567	D^7
2389	1257	1268	2358	2367	D^4
23·10·11	1257	1268	2458	2467	D^6
2345	1256	1278	1358	1367	D^4
23·12·13	1256	1278	3458	3467	D^6
248·14	1235	1568	2578	3567	D^7
259·14	1238	1578	2568	3678	D^7
349·14	1236	1567	2678	3568	D^7
358·14	1237	1678	2567	3578	D^7
4589	1357	1368	2356	2378	D^4
45·10·11	1357	1368	2456	2478	D^6
4567	1358	1367	1456	1478	D^6
89·12·13	2356	2378	3457	3468	D^6
10·11·12·13	2456	2478	3457	3468	D^4
6789	1457	1468	2358	2367	D^6
67·10·11	1457	1468	2458	2467	D^4
67·12·13	1456	1478	3458	3467	D^4

H-transformations of D^7

159·13	1245	1348	2347	4578	D^6
15·10·12	1248	1345	2346	4568	D^6
16·10·13	1246	1347	2348	4678	D^6
169·12	1247	1346	2345	4567	D^6
238·11	1237	1268	2358	2567	D^6
2347	1236	1278	1358	1567	D^6
248·14	1235	1368	2378	3567	D^6
27·11·14	1258	1378	2368	5678	D^6
34·11·14	1256	1367	2678	3568	D^6
378·14	1257	1678	2367	3578	D^6
478·11	1357	1568	2356	2578	D^6
569·10	1457	1468	2458	2467	D^6
56·12·13	1456	1478	3458	3467	D^6
9·10·12·13	2456	2478	3457	3468	D^6

therefore

$D^0 \sim 56D^4$, $D^4 \sim 2D^0+6D^4+24D^6$, $D^6 \sim 6D^4+6D^6+8D^7$, $D^7 \sim 14D^6$

And $[D^0] = \{D^0, D^4, D^6, D^7\}$

In these designs, D^4 has six identity H^2-transformations and D^6 three. That is, $D^4 \overset{2}{\sim} 6D^4$, $D^6 \overset{2}{\sim} 3D^6$

And $[D^i]^2 = \{D^i\}$ (i=0,4,6,7)

Example 4. Let v=9, k=4, b=18, r=8, λ=3 and

D^0: B_1(1234), B_2(1257), B_3(1268), B_4(1356), B_5(1379), B_6(1469), B_7(1478), B_8(1589), B_9(2358), B_{10}(2369), B_{11}(2459), B_{12}(2467), B_{13}(2789), B_{14}(3457), B_{15}(3489), B_{16}(3678), B_{17}(4568), B_{18}(5679)

D^4: B_1(1234), B_2(1239), B_3(1278), B_4(1356), B_5(1457), B_6(1468), B_7(1589), B_8(1679), B_9(2357), B_{10}(2467), B_{11}(2489), B_{12}(2568), B_{13}(2569), B_{14}(3458), B_{15}(3469), B_{16}(3678), B_{17}(3789), B_{18}(4579)

D_1^6: B_1(1234), B_2(1239), B_3(1256), B_4(1368), B_5(1458), B_6(1467), B_7(1579), B_8(1789), B_9(2357), B_{10}(2458), B_{11}(2479), B_{12}(2678), B_{13}(2689), B_{14}(3469), B_{15}(3478), B_{16}(3567), B_{17}(3589), B_{18}(4569)

D_2^6: B_1(1234), B_2(1239), B_3(1256), B_4(1368), B_5(1459), B_6(1467), B_7(1578), B_8(1789), B_9(2378), B_{10}(2458), B_{11}(2467), B_{12}(2579), B_{13}(2689), B_{14}(3458), B_{15}(3479), B_{16}(3567), B_{17}(3569), B_{18}(4689)

D^7: B_1(1234), B_2(1239), B_3(1268), B_4(1356), B_5(1458), B_6(1467), B_7(1579), B_8(1789), B_9(2378), B_{10}(2457), B_{11}(2459), B_{12}(2568), B_{13}(2679), B_{14}(3469), B_{15}(3478), B_{16}(3567), B_{17}(3589), B_{18}(4689)

D_1^8: B_1(1234), B_2(1268), B_3(1269), B_4(1356), B_5(1379), B_6(1458), B_7(1478), B_8(1579), B_9(2357), B_{10}(2378), B_{11}(2459), B_{12}(2467), B_{13}(2589), B_{14}(3456), B_{15}(3489), B_{16}(3689), B_{17}(4679), B_{18}(5678)

D_2^8: B_1(1234), B_2(1256), B_3(1268), B_4(1357), B_5(1369), B_6(1478), B_7(1489), B_8(1579), B_9(2378), B_{10}(2379), B_{11}(2459), B_{12}(2467), B_{13}(2589), B_{14}(3456), B_{15}(3458), B_{16}(3689), B_{17}(4679), B_{18}(5678)

D_3^8: B_1(1234), B_2(1256), B_3(1259), B_4(1378), B_5(1379), B_6(1457), B_7(1468), B_8(1689), B_9(2358), B_{10}(2367), B_{11}(2478), B_{12}(2489), B_{13}(2679), B_{14}(3456), B_{15}(3469), B_{16}(3589), B_{17}(4579), B_{18}(5678)

D^9: B_1(1234), B_2(1239), B_3(1268), B_4(1356), B_5(1458), B_6(1479), B_7(1579), B_8(1678), B_9(2378), B_{10}(2457), B_{11}(2467), B_{12}(2569), B_{13}(2589), B_{14}(3458), B_{15}(3469), B_{16}(3567), B_{17}(3789), B_{18}(4689)

H-transformations of D^0

•12·16·17	1237	1245	3468	5678	D^4
•16·13·16	1249	1346	2378	6789	D^4
•18·10·17	1239	1458	2346	5689	D^4
•18·13·14	1345	1289	5789	2347	D^4
•26·10·14	1457	1269	3469	2357	D^4
•26·13·17	1279	1456	2578	4689	D^4
•28·10·16	1259	1578	2367	3689	D^4
•34·11·15	1256	1368	2489	3459	D^4
•359·18	1238	1679	2568	3579	D^4
•35·12·15	1267	1389	2468	3479	D^4
•37·11·18	1248	1678	2569	4579	D^4
•45·11·12	1359	1367	2456	2479	D^4
•479·12	1358	1467	2356	2478	D^4
•47·15·18	1567	1348	4789	3569	D^4
•579·11	1378	1479	2359	2458	D^4
•68·14·16	1459	1689	3467	3578	D^4
•9·12·15·18	2348	2567	3589	4679	D^4
•10·13·14·17	2379	2689	3456	4578	D^4

H-transformations of D^4

•15·12·16	1245	1347	2368	5678	D^0
16·11·15	1248	1346	2349	4689	D_2^6
•17·12·15	1349	1258	5689	2346	D_1^6
•18·11·16	1249	1367	2348	2789	D_1^6
•23·14·18	1238	1279	3459	4578	D_2^6
•24·10·18	1236	1359	2479	4567	D_2^6
•26·10·17	1389	1246	4678	2379	D_2^6
•26·13·14	1269	1348	2359	4568	D_2^6
•34·10·14	1267	1358	2478	3456	D_3^8
•34·13·17	1378	1256	3569	2789	D_2^6
•36·13·18	1268	1478	2579	4569	D_2^6
•46·17·18	1368	1456	3579	4789	D_2^6
5678	1458	1467	1579	1689	D_2^6
•57·15·16	1459	1578	3467	3689	D_1^6
•58·11·12	1479	1567	2458	2689	D_1^6
67·11·12	1489	1568	2468	2589	D_2^6
68·15·16	1469	1678	3468	3679	D_2^6
•10·13·14·17	2456	2679	3478	3589	D_2^6

H-transformations of D_1^6

1345	1236	1245	1348	1568	D^7
•15·12·16	1248	1345	2367	5678	D^4
16·11·14	1247	1346	2349	4679	D^7
•18·12·14	1349	1278	6789	2346	D^4
•24·10·18	1238	1369	2459	4568	D^7
•269·18	1237	1469	2359	4567	D^7
•26·13·15	1269	1347	2389	4678	D_2^8
•27·10·15	1259	1379	2348	4578	D^7
34·12·16	1268	1356	2567	3678	D^7
•469·10	1367	1468	2358	2457	D_2^8
•479·13	1689	1357	2579	2368	D^7
•47·15·18	1378	1569	3468	4579	D_2^8
•58·14·16	1489	1578	3456	3679	D^4
58·15·17	1478	1589	3458	3789	D^7
•67·10·13	1457	1679	2468	2589	D^7
68·11·12	1479	1678	2467	2789	D^7
•9·13·15·18	2378	2569	3457	4689	D^7
14·15·16·17	3467	3489	3569	3578	D^7

H-transformations of D_2^6

•16·12·17	1247	1346	2359	5679	D^7
•18·12·14	1348	1279	5789	2345	D^7
•23·14·18	1235	1269	3489	4568	D^7
•25·11·16	1249	1359	2367	4567	D^7
25·12·15	1259	1349	2379	4579	D_2^8
•269·18	1237	1469	2389	4678	D^7
•27·10·15	1379	1258	4578	2349	D^7
•27·13·16	1289	1357	2369	5678	D^4
34·13·17	1268	1356	2569	3689	D^7
•369·14	1267	1456	2358	3478	D_2^8
•389·17	1569	1278	3789	2356	D_2^8
489·13	1378	1689	2368	2789	D^7
57·10·12	1458	1579	2459	2578	D^7
•57·11·13	1457	1589	2469	2678	D^7
•68·14·17	1478	1679	3456	3589	D^4
10·11·12·13	2457	2468	2589	2679	D^4
•10·13·15·16	2489	2568	3457	3679	D^7
11·12·15·16	2479	2567	3467	3579	D^7

H-transformations of D^7

1345	1236	1248	1345	1568	D_1^6
15·10·15	1245	1348	2347	4578	D_2^6
•18·10·17	1247	1389	2345	5789	D_1^8
239·13	1238	1269	2379	2678	D_2^6
•25·12·14	1349	1258	4568	2369	D_1^8
•269·18	1237	1469	2389	4678	D_1^8
•26·11·16	1249	1367	2359	4567	D_1^8
•28·12·16	1289	1379	2356	5678	D_1^6
•34·10·15	1256	1368	2478	3457	D_2^6
36·13·18	1267	1468	2689	4679	D_1^8
•37·10·18	1689	1257	4579	2468	D_2^6
•47·15·18	1357	1569	3468	4789	D_2^6
•58·14·16	1489	1578	3456	3679	D_1^8
58·15·17	1478	1589	3458	3789	D_1^8
67·10·13	1457	1679	2467	2579	D_1^8
•68·11·12	1479	1678	2456	2589	D_1^8
•9·11·16·18	2357	2489	3678	4569	D_2^6
14·15·16·17	3467	3489	3569	3578	D_1^8

H-transformations of D_1^8

1246	1236	1248	1345	1568	D^9
12·10·12	1238	1246	2347	2678	D_2^8
15·10·15	1237	1349	2348	3789	D^7
16·11·15	1245	1348	2349	4589	D^7
•24·11·15	1256	1368	2489	3459	D_2^8
•25·12·15	1267	1389	2468	3479	D^7
•279·14	1278	1468	2356	3457	D^7
•289·16	1689	1257	3579	2368	D^7
•37·11·18	1249	1678	2569	4578	D^9
38·11·17	1259	1679	2469	4579	D^7
•45·11·12	1359	1367	2456	2479	D^7
•46·10·12	1358	1456	2367	2478	D^7
•56·10·11	1378	1459	2379	2458	D_2^8
•78·14·16	1457	1789	3468	3569	D_2^8
78·17·18	1479	1578	4678	5679	D_2^8
•10·13·14·17	2358	2789	3467	4569	D^7
10·13·16·18	2389	2578	3678	5689	D^7
14·16·17·18	3469	3568	4567	6789	D_2^8

H-transformations of D_2^8

1357	1236	1248	1349	1689	D_1^8
139·12	1238	1246	2347	2678	D_1^8
149·15	1237	1345	2348	3578	D_1^8
17·11·15	1249	1348	2345	4589	D_1^8
•269·14	1456	1278	3478	2356	D_1^6
•26·13·17	1258	1467	2569	4789	D_1^6
•289·16	1257	1569	2368	3789	D_1^8
28·13·18	1259	1567	2568	5789	D_1^8
•34·12·15	1267	1358	2468	3457	D_2^6
•35·11·15	1269	1368	2458	3459	D_3^8
•45·11·12	1359	1367	2457	2469	D_2^6
•479·11	1378	1459	2357	2489	D_1^8
•579·12	1389	1469	2367	2478	D_1^8
•68·14 16	1457	1789	3468	3569	D_1^8
68·17·18	1479	1578	4678	5679	D_3^8
•9·13·14·17	2358	2789	3467	4569	D_1^8
9·13·16·18	2389	2578	3678	5689	D_2^6
14·16·17·18	3469	3568	4567	6789	D_1^8

H-transformations of D_3^8

1356	1239	1245	1347	1579	D_2^8
139·12	1235	1249	2348	2589	D_2^8
15·10·15	1237	1349	2346	3679	D_2^8
17·12·15	1248	1346	2349	4689	D_2^8
•24·11·14	1356	1278	3478	2456	D^4
•24·13·16	1267	1358	2569	3789	D_2^8
•28·11·17	1268	1569	2457	4789	D_2^8
28·13·18	1269	1568	2567	6789	D_2^8
•36·10·15	1257	1459	2369	3467	D_2^8
•379·15	1258	1469	2359	3468	D_2^8
•48·14·17	1368	1789	3457	4569	D_2^8
48·16·18	1389	1678	3578	5689	D_2^8
•569·12	1357	1479	2389	2458	D_2^8
•57·10·12	1367	1489	2379	2468	D_2^8
•679·10	1458	1467	2357	2368	D^4
•11·13·14·16	2467	2789	3458	3569	D_2^8
11·13·17·18	2479	2678	4578	5679	D_2^8
14·16·17·18	3459	3568	4567	5789	D_2^8

H-transformations of D^9

1345	1236	1248	1345	1568	D_1^8
139·11	1238	1246	2347	2678	D_1^8
•14·13·18	1235	1346	2489	5689	D_1^8
16·11·15	1247	1349	2346	4679	D_1^8
•16·12·16	1249	1347	2356	5679	D_1^8
•25·10·17	1389	1245	4578	2379	D_1^8
27·13·17	1259	1379	2389	5789	D_1^8
•28·13·16	1289	1367	2359	5678	D_1^8
35·13·18	1258	1468	2689	4589	D_1^8
•369·15	1278	1469	2368	3479	D_1^8
•37·10·18	1689	1257	4579	2468	D_1^8
•459·11	1358	1456	2367	2478	D_1^8
57·10·13	1457	1589	2458	2579	D_1^8
•78·14·15	1578	1679	3459	3468	D_1^8
78·16·17	1567	1789	3579	3678	D_1^8
•11·12·14·17	2456	2679	3478	3589	D_1^8
11·12·15·16	2469	2567	3467	3569	D_1^8
14·15·16·17	3456	3489	3578	3679	D_1^8

therefore

$D^0 \sim 18D^4$, $D^4 \sim D^0+4D_1^6+12D_2^6+D_3^8$, $D_1^6 \sim 3D^4+12D^7+3D_2^8$, $D_2^6 \sim 3D^4+12D^7+3D_2^8$, $D^7 \sim 2D_1^6+6D_2^6+10D_1^8$, $D_1^8 \sim 10D^7+6D_2^8+2D^9$, $D_2^8 \sim D_1^6+3D_2^6+12D_1^8+2D_3^8$, $D_3^8 \sim 2D^4+16D_2^8$, $D^9 \sim 18D_1^8$

In case of *, an H-transformation has a conjugate H^2-transformation, therefore

$D^0 \overset{2}{\sim} 18D^4$, $D^4 \overset{2}{\sim} D^0+4D_1^6+8D_2^6+D_3^8$, $D_1^6 \overset{2}{\sim} 3D^4+6D^7+3D_2^8$, $D_2^6 \overset{2}{\sim} 2D^4+8D^7+2D_2^8$, $D^7 \overset{2}{\sim} D_1^6+4D_2^6+6D_1^8$, $D_1^8 \overset{2}{\sim} 6D^7+3D_2^8+D^9$, $D_2^8 \overset{2}{\sim} D_1^6+2D_2^6+6D_1^8+D_3^8$, $D_3^8 \overset{2}{\sim} 2D^4+8D_2^8$, $D^9 \overset{2}{\sim} 9D_1^8$

And $[D^0]^s = \{D^0, D^4, D_1^6, D_2^6, D^7, D_1^8, D_2^8, D_3^8, D^9\}$, $(s=1,2)$

Example 5. Let $v=10$, $k=4$, $b=15$, $r=6$, $\lambda=2$ and

D^1: B_1(0123), B_2(0145), B_3(0268), B_4(0379), B_5(0489), B_6(0567), B_7(1279), B_8(1368), B_9(1467), B_{10}(1589), B_{11}(2345), B_{12}(2469), B_{13}(2578), B_{14}(3478), B_{15}(3569)

D^2: B_1(0123), B_2(0145), B_3(0268), B_4(0379), B_5(0489), B_6(0567), B_7(1247), B_8(1368), B_9(1589), B_{10}(1679), B_{11}(2359), B_{12}(2469), B_{13}(2578), B_{14}(3456), B_{15}(3478)

D^3: B_1(0123), B_2(0145), B_3(0268), B_4(0367), B_5(0489), B_6(0579), B_7(1247), B_8(1389), B_9(1568), B_{10}(1679), B_{11}(2359), B_{12}(2469), B_{13}(2578), B_{14}(3456), B_{15}(3478)

H-transformations of D^1

12·12·15	0124	0135	2369	4569	D^2
12·13·14	0125	0134	2378	4578	D^2
139·14	0126	0238	1347	4678	D^2
13·10·15	0128	0236	1359	5689	D^2
149·12	0137	0239	1246	4679	D^2
14·10·13	0139	0237	1258	5789	D^2
157·14	0129	0348	1237	4789	D^2
158·12	0138	0249	1236	4689	D^2
15·10·11	0234	0189	4589	1235	D^2
167·15	0127	0356	1239	5679	D^2
168·13	0136	0257	1238	5678	D^2
169·11	0235	0167	4567	1234	D^2
239·13	0146	0258	1457	2678	D^2
23^8·11	0245	0168	2368	1345	D^2
23·10·12	0158	0246	1459	2689	D^2
247·11	0345	0179	2379	1245	D^2
249·15	0147	0359	1456	3679	D^2
24·10·14	0159	0347	1458	3789	D^2
257·13	0149	0458	1257	2789	D^2
258·15	0148	0459	1356	3689	D^2
267·12	0157	0456	1249	2679	D^2
268·14	0156	0457	1348	3678	D^2
3478	0368	0279	1379	1268	D^2
34·12·14	0269	0378	2468	3479	D^2
34·13·15	0278	0369	2568	3579	D^2
3579	0289	0468	1267	1479	D^2
35·11·15	0248	0689	2356	3459	D^2
367·10	0267	0568	1289	1579	D^2
36·11·14	0256	0678	2348	3457	D^2
4589	0389	0479	1367	1468	D^2
45·11·13	0349	0789	2357	2458	D^2
468·10	0367	0579	1389	1568	D^2
46·11·12	0357	0679	2349	2456	D^2
569·10	0589	0467	1567	1489	D^2
56·12·13	0469	0578	2489	2567	D^2
56·14·15	0478	0569	3489	3567	D^2
78·12·14	1269	1378	2479	3468	D^2
78·13·15	1278	1369	2579	3568	D^2
79·11·15	1247	1679	2359	3456	D^2
7·10·11·14	1259	1789	2347	3458	D^2
89·11·13	1346	1678	2358	2457	D^2
8·10·11·12	1358	1689	2346	2459	D^2
9·10·12·13	1469	1578	2467	2589	D^2
9·10·14·15	1478	1569	3467	3589	D^2
12·13·14·15	2569	2478	3578	3469	D^2

H-transformations of D^2

12·13·15	0125	0134	2378	4578	D^3
149·13	0139	0237	1258	5789	D^3
158·12	0138	0249	1236	4689	D^3
167·14	0127	0356	1234	4567	D^3
168·13	0136	0257	1238	5678	D^1
16·10·11	0235	0167	5679	1239	D^3
239·12	0158	0246	1459	2689	D^3
247·11	0147	0359	1245	2379	D^3
249·15	0159	0347	1458	3789	D^1
24·10·14	0345	0179	3679	1456	D^3
268·15	0156	0457	1348	3678	D^3
34·12·15	0269	0378	2468	3479	D^3
357·10	0248	0689	1267	1479	D^3
35·11·14	0289	0468	2356	3459	D^3
4689	0367	0579	1389	1568	D^3
56·12·13	0469	0578	2489	2567	D^3
78·13·14	1278	1346	2457	3568	D^3
79·11·15	1478	1259	3589	2347	D^3
7·10·11·14	1279	1467	2345	3569	D^1
8·10·11·13	1369	1678	2358	2579	D^3
9·10·14·15	1569	1789	3458	3467	D^3

H-transformations of D^3

12·13·15	0125	0134	2378	4578	D^2
149·13	0136	0237	1258	5678	D^2
168·13	0139	0257	1238	5789	D^2
249·15	0156	0347	1458	3678	D^2
268·15	0159	0457	1348	3789	D^2
357·10	0689	0248	1479	1267	D^2
35·11·14	0289	0468	2356	3459	D^2
4689	0567	0379	1589	1368	D^2
7·10·11·14	1467	1279	3569	2345	D^2

An H-transformation has always a conjugate H^2-transformation, therefore

$D^1 \overset{s}{\sim} 45D^2$, $D^2 \overset{s}{\sim} 3D^1+18D^3$, $D^3 \overset{s}{\sim} 9D^2$ (s=1,2), and

$[D^1]^s = \{D^1, D^2, D^3\}$, (s=1,2)

Example 6. Let v=10, k=5, b=18, r=9, λ=4 and

D: B_1(01234), B_2(01256), B_3(01378), B_4(01679), B_5(02478), B_6(02579), B_7(03469), B_8(03589), B_9(04568), B_{10}(12368), B_{11}(12589), B_{12}(13459), B_{13}(14567), B_{14}(14789), B_{15}(23457), B_{16}(23679), B_{17}(24689), B_{18}(35678)

D_1^1: B_1(01234), B_2(01235), B_3(01478), B_4(01679), B_5(02579), B_6(02678), B_7(03469), B_8(03568), B_9(04589), B_{10}(12489), B_{11}(12569), B_{12}(13578), B_{13}(13689), B_{14}(14567), B_{15}(23467), B_{16}(23789), B_{17}(24568), B_{18}(34579)

D_2^1: B_1(01234), B_2(01235), B_3(01478), B_4(01679), B_5(02579), B_6(02678), B_7(03469), B_8(03568), B_9(04589), B_{10}(12469), B_{11}(12589), B_{12}(13567), B_{13}(13789), B_{14}(14568), B_{15}(23478), B_{16}(23689), B_{17}(24567), B_{18}(34579)

D_3^1: B_1(01234), B_2(01235), B_3(01469), B_4(01478), B_5(02579), B_6(02678), B_7(03568), B_8(03679), B_9(04589), B_{10}(12568), B_{11}(12789), B_{12}(13579), B_{13}(13689), B_{14}(14567), B_{15}(23467), B_{16}(23489), B_{17}(24569), B_{18}(34578)

D^2: B_1(01234), B_2(01235), B_3(01467), B_4(01678), B_5(02578), B_6(02689), B_7(03489), B_8(03569), B_9(04579), B_{10}(12469), B_{11}(12579), B_{12}(13568), B_{13}(13789), B_{14}(14589), B_{15}(23478), B_{16}(23679), B_{17}(24568), B_{18}(34567)

D_1^3: B_1(01234), B_2(01235), B_3(01567), B_4(01678), B_5(02469), B_6(02589), B_7(03468), B_8(03789), B_9(04579), B_{10}(12478), B_{11}(12689), B_{12}(13479), B_{13}(13569), B_{14}(14589), B_{15}(23578), B_{16}(23679), B_{17}(24567), B_{18}(34568)

D_2^3: B_1(01234), B_2(01235), B_3(01467), B_4(01489), B_5(02579), B_6(02689), B_7(03568), B_8(03789), B_9(04567), B_{10}(12569), B_{11}(12678), B_{12}(13578), B_{13}(13679), B_{14}(14589), B_{15}(23468), B_{16}(23479), B_{17}(24578), B_{18}(34569)

D^5: B_1(01234), B_2(01235), B_3(01567), B_4(01678), B_5(02589), B_6(02689), B_7(03468), B_8(03479), B_9(04579), B_{10}(12469), B_{11}(12478), B_{12}(13569), B_{13}(13789), B_{14}(14589), B_{15}(23578), B_{16}(23679), B_{17}(24567), B_{18}(34568)

D^9: B_1(01234), B_2(01235), B_3(01567), B_4(01678), B_5(02589), B_6(02789), B_7(03469), B_8(03478), B_9(04569), B_{10}(12468), B_{11}(12479), B_{12}(13589), B_{13}(13689), B_{14}(14579), B_{15}(23567), B_{16}(23679), B_{17}(24568), B_{18}(34578)

$E=D^c$, $E_1^1=D_1^{1c}$, $E_2^1=D_2^{1c}$, $E^2=D^{2c}$, $E_1^3=D_1^{3c}$, $E_2^3=D_2^{3c}$, $E^5=D^{5c}$

D_3^1, D^9 are self-complementary, that is $D_3^{1c}\cong D_3^1$, $D^{9c}\cong D^9$

H-transformations of D

14·10·14	01236	01479	12346	16789	D_1^1
19·12·17	01345	02468	12349	45689	D_1^1
25·10·15	01268	02457	12356	23478	D_1^1
27·13·16	01456	02369	12567	34679	D_1^1
36·14·15	01789	02357	13478	24579	D_1^1
37·12·18	03678	01349	34569	13578	D_1^1
48·11·16	03679	01589	23589	12679	D_1^1
58·17·18	02489	03578	24678	35689	D_1^1
69·11·13	02589	04567	12579	14568	D_1^1

H-transformations of D_1^1

15·11·15	02347	01259	25679	12346	D_2^1
18·12·15	02346	01358	35678	12347	D^2
26·11·16	01256	02378	12359	26789	D
34·15·16	01467	01789	23478	23679	E^2
37·10·15	01489	03467	12478	23469	E^2
47·10·16	01469	03679	12789	23489	E_2^1
49·13·18	01689	04579	13679	34589	E^2
58·11·12	02569	03578	12579	13568	D^2
58·17·18	03579	02568	34568	24579	D^2
11·12·17·18	12568	13579	24569	34578	D_2^1

H-transformations of D_2^1

15·11·15	02347	01259	25789	12348	D_1^1
18·12·15	02348	01358	35678	12347	D_1^1
37·10·15	03478	01469	23469	12478	E_2^3
37·14·18	01468	03479	14578	34569	E_1^1
39·10·17	01489	04576	12467	24569	E_1^1
56·12·13	02567	02789	13579	13678	D_1^3
58·11·12	02589	03567	12579	13568	D_1^3
68·11·13	02568	03678	12789	13589	D_3^1
79·15·17	03489	04569	23467	24578	D_1^1
10·14·15·18	12468	14569	23479	34578	D_1^1

H-transformations of D_3^1

34·15·16	01467	01489	23469	23478	D^5
36·13·15	01689	02467	13469	23678	D_2^1
36·11·16	01679	03469	12489	23789	D_2^1
46·13·16	01678	02478	13489	23689	D_2^1
48·11·15	01789	03467	12478	23679	D_2^1
57·10·12	03579	02568	13568	12579	E^5
57·17·18	02569	03578	24579	34566	E_2^1
59·10·14	02589	04579	12567	14568	E_2^1
79·12·14	03589	04568	13567	14579	E_2^1
10·12·17·18	12569	13578	24568	34579	E_2^1

H-transformations of D^2

14·10·13	01246	01378	12349	16789	D_1^1
19·10·18	01249	03457	12346	45679	D_1^1
19·14·15	02347	01459	45789	12348	E_1^1
28·14·17	01359	02356	12458	45689	E_1^1
37·10·15	01469	03478	12467	23489	E_1^1
37·14·18	03467	01489	34589	14567	D_1^1
49·13·18	01789	04567	13678	34579	D_1^3
56·12·13	02568	02789	13578	13689	D_1^3
58·11·12	02579	03568	12578	13569	E_1^3
68·11·13	02569	03689	12789	13579	E_1^3
10·14·15·18	12489	14569	23467	34578	E_2^3

H-transformations of D_1^3

1458	01246	01378	02349	06789	D_2^1
14·11·12	01347	01268	16789	12349	D_2^1
15·14·18	01249	02346	13458	45689.	D^2
29·13·17	01359	02457	12356	45679	E^2
29·14·15	02357	01459	45789	12358	E^2
36·13·15	01569	02578	13567	23589	E^2
36·14·17	02567	01589	24589	14567	E^2
48·14·18	01789	03678	14568	34589	D^2
57·10·12	02468	03469	12479	13478	D^2
58·11·12	02689	03479	12469	13789	D^5
78·10·11	03478	03689	12468	12789	D^2
13·14·15·17	13589	14569	23567	24578	E^5

H-transformations of D_2^3

1368	01246	01347	02389	06789	E_2^1
14·11·13	01248	01349	12367	16789	E_2^1
16·11·18	02349	01268	26789	12347	E^2
18·13·15	02348	01379	36789	12346	E^2
34·15·16	01468	01479	23467	23489	E_2^1
36·13·15	01679	02468	13467	23689	E^2
38·11·16	01678	03479	12467	23789	E^2
46·13·16	01689	02489	13479	23679	E^2
48·11·15	01789	03489	12468	23678	E^2
57·10·12	02569	03578	12579	13568	D^5
57·17·18	02578	03569	24579	34568	D^5
10·12·17·18	13569	12578	34578	24569	D^5

H-transformations of D^5

1468	01347	01268	06789	02349	D_3^1
14·10·13	01246	01378	12349	16789	D_1^3
16·11·16	01248	02369	12347	26789	D_1^3
17·13·16	01348	02346	12379	36789	E_1^3
29·12·17	01359	02457	12356	45679	D_2^3
29·14·15	02357	01459	45789	12358	D_2^3
35·12·15	01569	02578	13567	23589	D_2^3
35·14·17	02567	01589	24589	14567	D_2^3
47·10·16	01468	03678	12679	23469	E_1^3
48·11·16	01478	03679	12678	23479	D_1^3
67·11·13	02468	03689	12789	13478	E_1^3
68·10·13	02469	03789	12689	13479	D_1^3
78·10·11	03469	03478	12468	12479	E_1^3
12·14·15·17	13589	14569	23567	24578	D^9

H-transformations of D^9

1467	01346	01278	06789	02349	E^5
14·11·13	01247	01368	12349	16789	E^5
16·10·16	01248	02379	12346	25789	E^5
18·13·16	01348	02347	12369	36789	E^5
23·17·18	01256	01357	23456	45678	D^5
25·14·18	01259	02358	13457	45789	D^5
29·12·17	01359	02456	12358	45689	D^5
29·14·15	02356	01459	45679	12357	D^5
35·12·15	02567	01589	23589	13567	D^5
35·14·17	01579	02568	14567	24589	D^5
39·12·18	01569	04567	13578	34589	D^5
47·10·16	01468	03679	12678	23469	E^5
48·11·16	01478	03678	12679	23479	E^5
59·15·18	02569	04589	23578	34567	D^5
67·11·13	02479	03689	12789	13469	E^5
68·10·13	02478	03789	12689	13468	E^5
78·10·11	03468	03479	12469	12478	E^5
12·14·15·17	13579	14589	23568	24567	D^5

An H-transformation has always a conjugate H^2-transformation, but these designs don't have H^3-transformations. Therefore $D \overset{s}{\sim} 9D_1^1$, $E \overset{s}{\sim} 9E_1^1$, $D_1^1 \overset{s}{\sim} D+2D_2^1+E_2^1+3D^2+3E^2$, $E_1^1 \overset{s}{\sim} E+D_2^1+2E_2^1+3D^2+3E^2$, $D_2^1 \overset{s}{\sim} 4D_1^1+2E_1^1+D_3^1+2D_1^3+E_2^3$, $E_2^1 \overset{s}{\sim} 2D_1^1+4E_1^1+D_3^1+2E_1^3+D_2^3$, $D_3^1 \overset{s}{\sim} 4D_2^1+4E_2^1+D^5+E^5$, $D^2 \overset{s}{\sim} 3D_1^1+3E_1^1+2D_1^3+2E_1^3+E_2^3$, $E^2 \overset{s}{\sim} 3D_1^1+3E_1^1+2D_1^3+2E_1^3+D_2^3$, $D_1^3 \overset{s}{\sim} 2D_2^1+4D^2+4E^2+D^5+E^5$, $E_1^3 \overset{s}{\sim} 2E_2^1+4D^2+4E^2+D^5+E^5$, $D_2^3 \overset{s}{\sim} 3E_2^1+6E^2+3D^5$, $E_2^3 \overset{s}{\sim} 3D_2^1+6D^2+3E^5$, $D^5 \overset{s}{\sim} D_3^1+4D_1^3+4E_1^3+4D_2^3+D^9$, $E^5 \overset{s}{\sim} D_3^1+4D^3+4E_1^3+4E_2^3+D^9$, $D^9 \overset{s}{\sim} 9D^5+9E^5$ (s=1,2)

And $[D]^s=\{D, E, D_1^1, E_1^1, D_2^1, E_2^1, D_3^1, D^2, E^2, D_1^3, E_1^3, D_2^3, E_2^3, D^5, E^5, D^9\}$, (s=1,2)

Example 7. Let v=b=16, k=r=6, λ=2 and

D_1: B_1(123456), B_2(12789·10), B_3(137·11·12·13), B_4(148·11·14·15), B_5(159·12·14·16), B_6(16·10·13·15·16), B_7(237·14·15·16), B_8(248·12·13·16), B_9(259·11·13·15), B_{10}(26·10·11·12·14), B_{11}(349·10·11·16), B_{12}(358·10·12·15), B_{13}(3689·13·14), B_{14}(457·10·13·14), B_{15}(4679·12·15), B_{16}(5678·11·16)

D_2: B_1(123456), B_2(12789·10), B_3(137·11·12·13), B_4(148·11·14·15), B_5(159·12·14·16), B_6(16·10·13·15·16), B_7(237·14·15·16), B_8(248·12·13·16), B_9(259·11·13·15), B_{10}(26·10·11·12·14), B_{11}(349·10·11·16), B_{12}(358·10·13·14), B_{13}(3689·12·15), B_{14}(457·10·12·15), B_{15}(4679·13·14), B_{16}(5678·11·16)

D_3: B_1(123456), B_2(12789·10), B_3(137·11·12·13), B_4(148·11·14·15), B_5(159·12·14·16), B_6(16·10·13·15·16), B_7(237·14·15·16), B_8(248·12·13·16), B_9(259·11·13·15), B_{10}(26·10·11·12·14), B_{11}(349·10·12·15), B_{12}(358·10·13·14), B_{13}(3689·11·16), B_{14}(457·10·11·16), B_{15}(4679·13·14), B_{16}(5678·12·15)

H-transformations of D_1

12·11·16	12349·10	125678	3456·11·16	789·10·11·16	D_2
12·12·15	12358·10	124679	3456·12·15	789·10·12·15	D_2
12·13·14	123689	12457·10	3456·13·14	789·10·13·14	D_2
138·16	1234·12·13	13567·11	24568·16	78·11·12·13·16	D_2
139·15	1235·11·13	13467·12	24569·15	79·11·12·13·15	D_2
13·10·14	1236·11·12	13457·13	2456·10·14	7·10·11·12·13·14	D_2
147·16	1234·14·15	14568·11	23567·16	78·11·14·15·16	D_2
149·13	1245·11·15	13468·14	23569·13	89·11·13·14·15	D_2
14·10·12	1246·11·14	13458·15	2356·10·12	8·10·11·12·14·15	D_2
157·15	1235·14·16	14569·12	23467·15	79·12·14·15·16	D_2
158·13	1245·12·16	13569·14	23468·13	89·12·13·14·16	D_2
15·10·11	1256·12·14	13459·16	2346·10·11	9·10·11·12·14·16	D_2
167·14	1236·15·16	1456·10·13	23457·14	7·10·13·14·15·16	D_2
168·12	1246·13·16	1356·10·15	23458·12	8·10·12·13·15·16	D_2
169·11	1256·13·15	1346·10·16	23459·11	9·10·11·13·15·16	D_2
238·11	1278·12·13	1379·10·11	2489·10·16	34·11·12·13·16	D_2
239·12	1279·11·13	1378·10·12	2589·10·15	35·11·12·13·15	D_2
23·10·13	127·10·11·12	13789·13	2689·10·14	36·11·12·13·14	D_2
247·11	1278·14·15	1489·10·11	2379·10·16	34·11·14·15·16	D_2
249·14	1289·11·15	1478·10·14	2579·10·13	45·11·13·14·15	D_2
24·10·15	128·10·11·14	14789·15	2679·10·12	46·11·12·14·15	D_2
257·12	1279·14·16	1589·10·12	2378·10·15	35·12·14·15·16	D_2
258·14	1289·12·16	1579·10·14	2478·10·13	45·12·13·14·16	D_2
25·10·16	129·10·12·14	15789·16	2678·10·11	56·11·12·14·16	D_2
267·13	127·10·15·16	1689·10·13	23789·14	36·13·14·15·16	D_2
268·15	128·10·13·16	1679·10·15	24789·12	46·12·13·15·16	D_2
269·16	129·10·13·15	1678·10·16	25789·11	56·11·13·15·16	D_2
3478	137·11·14·15	148·11·12·13	237·12·13·16	248·14·15·16	D_2
34·12·14	138·11·12·15	147·11·13·14	357·10·12·13	458·10·14·15	D_2
34·13·15	138·11·13·14	147·11·12·15	3679·12·13	4689·14·15	D_2
3579	137·12·14·16	159·11·12·13	237·11·13·15	259·14·15·16	D_2
35·11·14	139·11·12·16	157·12·13·14	347·10·11·13	459·10·14·16	D_2
35·13·16	139·12·13·14	157·11·12·16	3678·11·13	5689·14·16	D_2
367·10	137·13·15·16	16·10·11·12·13	237·11·12·14	25·10·14·15·16	D_2
36·11·15	13·10·11·13·16	167·12·13·15	3479·11·12	469·10·15·16	D_2
36·12·16	13·10·12·13·15	167·11·13·16	3578·11·12	568·10·15·16	D_2
4589	148·12·14·16	159·11·14·15	248·11·13·15	259·12·13·16	D_2
45·11·12	149·11·14·16	158·12·14·15	348·10·11·15	359·10·12·16	D_2
45·15·16	149·12·14·15	158·11·14·16	4678·11·15	5679·12·16	D_2
468·10	148·13·15·16	16·10·11·14·15	248·11·12·14	26·10·12·13·16	D_2
46·11·13	14·10·11·15·16	168·13·14·15	3489·11·14	369·10·13·16	D_2
46·14·16	14·10·13·14·15	168·11·15·16	4578·11·14	567·10·13·16	D_2
569·10	159·13·15·16	16·10·12·14·16	259·11·12·14	26·10·11·13·15	D_2
56·12·13	15·10·12·15·16	169·13·14·16	3589·12·14	368·10·13·15	D_2
56·14·15	15·10·13·14·16	169·12·15·16	4579·12·14	467·10·13·15	D_2
78·12·14	238·12·15·16	247·13·14·16	357·10·14·15	458·10·12·13	D_2
78·13·15	238·13·14·16	247·12·15·16	3679·14·15	4689·12·13	D_2
79·11·14	239·11·15·16	257·13·14·15	347·10·14·16	459·10·11·13	D_2
79·13·16	239·13·14·15	257·11·15·16	3678·14·16	5689·11·13	D_2
7·10·11·15	23·10·11·14·16	267·12·14·15	3479·15·16	469·10·11·12	D_2
7·10·12·16	23·10·12·14·15	267·11·14·16	3578·15·16	568·10·11·12	D_2
89·11·12	249·11·13·16	258·12·13·15	348·10·12·16	359·10·11·15	D_2
89·15·16	249·12·13·15	258·11·13·16	4678·12·16	5679·11·15	D_2
8·10·11·13	24·10·11·12·16	268·12·13·14	3489·13·16	369·10·11·14	D_2
8·10·14·16	24·10·12·13·14	268·11·12·16	4578·13·16	567·10·11·14	D_2
9 10 12 13	25·10·11·12·15	269·11·13·14	3589·13·15	368·10·12·14	D_2
9·10·14·15	25·10·11·13·14	269·11·12·15	4579·13·15	467·10·12·14	D_2
11·12·15·16	349·10·12·15	358·10·11·16	4679·11·16	5678·12·15	D_2
11·13·14·16	349·10·13·14	3689·11·16	457·10·11·16	5678·13·14	D_2
12·13·14·15	358·10·13·14	3689·12·15	457·10·12·15	4679·13·14	D_2

H-transformations of D_2

12·11·16	12349·10	125678	3456·11·16	789·10·11·16	D_1
12·12·15	12358·10	124679	3456·13·14	789·10·13·14	D_3
12·13·14	123689	12457·10	3456·12·15	789·10·12·15	D_3
138·16	1234·12·13	13567·11	24568·16	78·11·12·13·16	D_3
147·16	1234·14·15	14568·11	23567·16	78·11·14·15·16	D_3
15·10·11	1256·12·14	13459·16	2346·10·11	9·10·11·12·14·16	D_3
169·11	1256·13·15	1346·10·16	23459·11	9·10·11·13·15·16	D_3
238·11	1278·12·13	1379·10·11	2489·10·16	34·11·12·13·16	D_3
247·11	1278·14·15	1489·10·11	2379·10·16	34·11·14·15·16	D_3
25·10·16	129·10·12·14	15789·16	2678·10·11	56·11·12·14·16	D_3
269·16	129·10·13·15	1678·10·16	25789·11	56·11·13·15·16	D_3
3478	137·11·14·15	148·11·12·13	237·12·13·16	248·14·15·16	D_1
34·12·14	138·11·13·14	147·11·12·15	357·10·12·13	458·10·14·15	D_3
34·13·15	138·11·12·15	147·11·13·14	3679·12·13	4689·14·15	D_3
3579	137·12·14·16	159·11·12·13	237·11·13·15	259·14·15·16	D_3
367·10	137·13·15·16	16·10·11·12·13	237·11·12·14	26·10·14·15·16	D_3
4589	148·12·14·16	159·11·14·15	248·11·13·15	259·12·13·16	D_3
468·10	148·13·15·16	16·10·11·14·15	248·11·12·14	26·10·12·13·16	D_3
569·10	159·13·15·16	16·10·12·14·16	259·11·12·14	26·10·11·13·15	D_1
56·12·13	15·10·13·14·16	169·12·15·16	3589·12·14	368·10·13·15	D_3
56·14·15	15·10·12·15·16	169·13·14·16	4579·12·14	467·10·13·15	D_3
78·12·14	238·13·14·16	247·12·15·16	357·10·14·15	458·10·12·13	D_3
78·13·15	238·12·15·16	247·13·14·16	3679·14·15	4689·12·13	D_3
9·10·12·13	25·10·11·13·14	269·11·12·15	3589·13·15	368·10·12·14	D_3
9·10·14·15	25·10·11·12·15	269·11·13·14	4579·13·15	467·10·12·14	D_3
11·12·15·16	349·10·13·14	358·10·11·16	4679·11·16	5678·13·14	D_3
11·13·14·16	349·10·12·15	3689·11·16	457·10·11·16	5678·12·15	D_3
12·13·14·15	358·10·12·15	3689·13·14	457·10·13·14	4679·12·15	D_1

H-transformations of D_3

12·11·16	12349·10	125678	3456·12·15	789·10·12·15	D_2
12·12·15	12358·10	124679	3456·13·14	789·10·13·14	D_2
12·13·14	123689	12457·10	3456·11·16	789·10·11·16	D_2
3478	137·11·14·15	148·11·12·13	237·12·13·16	248·14·15·16	D_2
3579	137·12·14·16	159·11·12·13	237·11·13·15	259·14·15·16	D_2
367·10	137·13·15·16	16·10·11·12·13	237·11·12·14	26·10·14·15·16	D_2
4589	148·12·14·16	159·11·14·15	248·11·13·15	259·12·13·16	D_2
468·10	148·13·15·16	16·10·11·14·15	248·11·12·14	26·10·12·13·16	D_2
569·10	159·13·15·16	16·10·12·14·16	259·11·12·14	26·10·11·13·15	D_2
11·12·15·16	349·10·13·14	358·10·12·15	4679·12·15	5678·13·14	D_2
11·13·14·16	349·10·11·16	3689·12·15	457·10·12·15	5678·11·16	D_2
12·13·14·15	358·10·11·16	3689·13·14	457·10·13·14	4679·11·16	D_2

therefore

$D_1 \overset{2}{\sim} 60D_2$, $D_2 \overset{2}{\sim} 4D_1+24D_3$, $D_3 \overset{2}{\sim} 12D_2$

and $[D_1]^2=\{D_1, D_2, D_3\}$

Example 8. Let v=b=15, k=r=7, λ=3 and

D_1: B_1(1234567), B_2(12389·10·11), B_3(123·12·13·14·15)
B_4(14589·12·13), B_5(145·10·11·14·15) B_6(16789·14·15),
B_7(167·10·11·12·13) B_8(2468·10·12·14), B_9(2478·11·13·15),
B_{10}(2569·11·12·15), B_{11}(2579·10·13·14), B_{12}(3469·11·13·14),
B_{13}(3479·10·12·15), B_{14}(3568·10·13·15), B_{15}(3578·11·12·14)

D_2: B_1(1234567), B_2(12389·10·11), B_3(123·12·13·14·15),
B_4(14589·12·13), B_5(145·10·11·14·15), B_6(1678·10·12·14),
B_7(1679·11·13·15), B_8(24689·14·15), B_9(246·10·11·12·13),
B_{10}(2578·11·13·14), B_{11}(2579·10·12·15), B_{12}(3478·10·13·15),
B_{13}(3479·11·12·14), B_{14}(3568·11·12·15), B_{15}(3569·10·13·14)

D_3: B_1(1234567), B_2(12389·10·11), B_3(123·12·13·14·15),
B_4(14589·12·13), B_5(145·10·11·14·15), B_6(16789·14·15),
B_7(167·10·11·12·13), B_8(2468·10·12·14), B_9(2469·11·13·15),
B_{10}(2578·11·12·15), B_{11}(2579·10·13·14), B_{12}(3478·11·13·14),
B_{13}(3479·10·12·15), B_{14}(3568·10·13·15), B_{15}(3569·11·12·14)

D_4: B_1(123456), B_2(12389·10·11), B_3(123·12·13·14·15),
B_4(14589·12·13), B_5(145·10·11·14·15), B_6(16789·14·15),
B_7(167·10·11·12·13), B_8(2468·10·12·14), B_9(2469·11·13·15),
B_{10}(2578·11·12·15), B_{11}(2579·10·13·14), B_{12}(3478·10·13·15),
B_{13}(3479·11·12·14), B_{14}(3568·11·13·14), B_{15}(3569·10·12·15)

D_5: B_1(123456), B_2(12389·10·11), B_3(123·12·13·14·15),
B_4(14589·12·13), B_5(145·10·11·14·15), B_6(16789·14·15),
B_7(167·10·11·12·13), B_8(2468·10·12·14), B_9(2469·11·13·15),
B_{10}(2578·10·13·15), B_{11}(2579·11·12·14), B_{12}(3478·11·12·15),
B_{13}(3479·10·13·14), B_{14}(3568·11·13·14), B_{15}(3569·10·12·15)

H-transformations of D_1

1247	1234589	12367·10·11	14567·12·13	189·10·11·12·13	D_3
1256	12345·10·11	1236789	14567·14·15	189·10·11·14·15	D_3
1346	12345·12·13	12367·14·15	1456789	189·12·13·14·15	D_3
1357	12345·14·15	12367·12·13	14567·10·11	1·10·11·12·13·14·15	D_3
2345	12389·12·13	123·10·11·14·15	14589·10·11	145·12·13·14·15	D_3
2367	12389·14·15	123·10·11·12·13	14589·12·13	145·10·11·14·15	D_3
4567	14589·14·15	145·10·11·12·13	16789·12·13	167·10·11·14·15	D_3
89·10·11	2468·11·12·15	2478·10·13·14	2569·10·12·14	2579·11·13·15	D_3
89·12·13	2468·11·13·14	2478·10·12·15	3469·10·12·14	3479·11·13·15	D_3
89·14·15	2468·10·13·15	2478·11·12·14	3568·10·12·14	3578·11·13·15	D_3
8·10·12·14	2469·11·12·14	2568·10·12·15	3468·10·13·14	3569·11·13·15	D_3
8·10·13·15	2469·10·12·15	2568·11·12·14	3478·10·12·14	3579·11·12·15	D_3
8·11·12·15	2469·10·13·14	2578·10·12·14	3468·11·12·14	3579·11·13·14	D_3
8·11·13·14	2478·10·13·14	2569·10·12·14	3468·10·12·15	3579·10·13·15	D_3
9·10·12·15	2469·11·13·15	2578·11·12·15	3478·11·13·14	3569·11·12·14	D_3
9·10·13·14	2479·11·12·15	2568·11·13·15	3478·10·13·15	3569·10·12·15	D_3
9·11·12·14	2479·11·13·14	2578·10·13·15	3468·11·13·15	3569·10·13·14	D_3
9·11·13·15	2479·10·13·15	2578·11·13·14	3478·11·12·15	3579·10·12·14	D_3
10·11·12·13	2569·11·13·14	2579·10·12·15	3469·11·12·15	3479·10·13·14	D_3
10·11·14·15	2569·10·13·15	2579·11·12·14	3568·11·12·15	3578·10·13·14	D_3
12·13·14·15	3469·10·13·15	3479·11·12·14	3568·11·13·14	3578·10·12·15	D_3

H-transformations of D_2

2345	12389·12·13	123·10·11·14·15	14589·10·11	145·12·13·14·15	D_3
2367	1238·10·12·14	1239·11·13·15	16789·10·11	167·12·13·14·15	D_3
2389	12389·14·15	123·10·11·12·13	24689·10·11	246·12·13·14·15	D_3
23·10·11	1238·11·13·14	1239·10·12·15	25789·10·11	257·12·13·14·15	D_3
23·12·13	1238·10·13·15	1239·11·12·14	34789·10·11	347·12·13·14·15	D_3
23·14·15	1238·11·12·15	1239·10·13·14	35689·10·11	356·12·13·14·15	D_3
4567	1458·10·12·14	1459·11·13·15	16789·12·13	167·10·11·14·15	D_3
4589	14589·14·15	145·10·11·12·13	24689·12·13	246·10·11·14·15	D_3
45·10·11	1458·11·13·14	1459·10·12·15	25789·12·13	257·10·11·14·15	D_3
45·12·13	1458·10·13·15	1459·11·12·14	34789·12·13	347·10·11·14·15	D_3
45·14·15	1458·11·12·15	1459·10·13·14	35689·12·13	356·10·11·14·15	D_3
6789	16789·14·15	167·10·11·12·13	2468·10·12·14	2469·11·13·15	D_3
67·10·11	1678·11·13·14	1679·10·12·15	2578·10·12·14	2579·11·13·15	D_3
67·12·13	1678·10·13·15	1679·11·12·14	3478·10·12·14	3479·11·13·15	D_3
67·14·15	1678·11·12·15	1679·10·13·14	3568·10·12·14	3569·11·13·15	D_3
89·10·11	2468·11·13·14	2469·10·12·15	25789·14·15	257·10·11·12·13	D_3
89·12·13	2468·10·13·15	2469·11·12·14	34789·14·15	347·10·11·12·13	D_3
89·14·15	2468·11·12·15	2469·10·13·14	35689·14·15	356·10·11·12·13	D_3
10·11·12·13	2578·10·13·15	2579·11·12·14	3478·11·13·14	3479·10·12·15	D_3
10·11·14·15	2578·11·12·15	2579·10·13·14	3568·11·13·14	3569·10·12·15	D_3
12·13·14·15	3478·11·12·15	3479·10·13·14	3568·10·13·15	3569·11·12·14	D_3

H-transformations of D_3

1247	1234589	12367·10·11	14567·12·13	189·10·11·12·13	D_1
1256	12345·10·11	1236789	14567·14·15	189·10·11·14·15	D_1
1346	12345·12·13	12367·14·15	1456789	189·12·13·14·15	D_1
1357	12345·14·15	12367·12·13	14567·10·11	1·10·11·12·13·14·15	D_1
2345	12389·12·13	123·10·11·14·15	14589·10·11	145·12·13·14·15	D_4
2367	12389·14·15	123·10·11·12·13	14589·12·13	145·10·11·14·15	D_4
2389	1238·10·12·14	1239·11·13·15	24689·10·11	246·12·13·14·15	D_2
23·10·11	1238·11·12·15	1239·10·13·14	25789·10·11	257·12·13·14·15	D_2
23·12·13	1238·11·13·14	1239·10·12·15	34789·10·11	347·12·13·14·15	D_2
23·14·15	1238·10·13·15	1239·11·12·14	35689·10·11	356·12·13·14·15	D_2
4567	14589·14·15	145·10·11·12·13	16789·12·13	167·10·11·14·15	D_4
4589	1458·10·12·14	1459·11·13·15	24689·12·13	246·10·11·14·15	D_2
45·10·11	1458·11·12·15	1459·10·13·14	25789·12·13	257·10·11·14·15	D_2
45·12·13	1458·11·13·14	1459·10·12·15	34789·12·13	347·10·11·14·15	D_2
45·14·15	1458·10·13·15	1459·11·12·14	35689·12·13	356·10·11·14·15	D_2
6789	1678·10·12·14	1679·11·13·15	24689·14·15	246·10·11·12·13	D_2
67·10·11	1678·11·12·15	1679·10·13·14	25789·14·15	257·10·11·12·13	D_2
67·12·13	1678·11·13·14	1679·10·12·15	34789·14·15	347·10·11·12·13	D_2
67·14·15	1678·10·13·15	1679·11·12·14	35689·14·15	356·10·11·12·13	D_2
89·10·11	2468·11·12·15	2469·10·13·14	2578·10·12·14	2579·11·13·15	D_4
89·12·13	2468·11·13·14	2469·10·12·15	3478·10·12·14	3479·11·13·15	D_4
89·14·15	2468·10·13·15	2469·11·12·14	3568·10·12·14	3569·11·13·15	D_4
8·10·12·14	2478·11·12·14	2568·10·12·15	3468·10·13·14	3578·11·13·15	D_1
8·10·13·15	2478·10·12·15	2568·11·12·14	3469·10·12·14	3579·11·12·15	D_1
8·11·12·15	2478·10·13·14	2569·10·12·14	3468·11·12·14	3579·11·13·14	D_1
8·11·13·14	2479·10·12·14	2568·10·13·14	3468·10·12·15	3579·10·13·15	D_1
9·10·12·15	2478·11·13·15	2569·11·12·15	3469·11·13·14	3578·11·12·14	D_1
9·10·13·14	2479·11·12·15	2568·11·13·15	3469·10·13·15	3578·10·12·15	D_1
9·11·12·14	2479·11·13·14	2569·10·13·15	3468·11·13·15	3578·10·13·14	D_1
9·11·13·15	2479·10·13·15	2569·11·13·14	3469·11·12·15	3579·10·12·14	D_1
10·11·12·13	2578·11·13·14	2579·10·12·15	3478·11·12·15	3479·10·13·14	D_4
10·11·14·15	2578·10·13·15	2579·11·12·14	3568·11·12·15	3569·10·13·14	D_4
12·13·14·15	3478·10·13·15	3479·11·12·14	3568·11·13·14	3569·10·12·15	D_4

H-transformations of D_4

1247	1234589	12367·10·11	14567·12·13	189·10·11·12·13	D_3
1256	12345·10·11	1236789	14567·14·15	189·10·11·14·15	D_3
1346	12345·12·13	12367·14·15	1456789	189·12·13·14·15	D_3
1357	12345·14·15	12367·12·13	14567·10·11	1·10·11·12·13·14·15	D_3
168·15	124678·14	135679·15	23456·10·12	689·10·12·14·15	D_3
169·14	124679·15	135678·14	23456·11·13	689·11·13·14·15	D_3
16·10·13	125678·15	134679·14	23457·11·12	789·11·12·14·15	D_3
16·11·12	125679·14	134678·15	23457·10·13	789·10·13·14·15	D_3
178·14	12467·10·12	13567·11·13	234568·14	68·10·11·12·13·14	D_3
179·15	12467·11·13	13567·10·13	234569·15	69·10·11·12·13·15	D_3
17·10·12	12567·11·12	13467·10·13	234578·15	78·10·11·12·13·15	D_3
17·11·13	12567·10·13	13467·11·12	234579·14	79·10·11·12·13·14	D_3
2345	12389·12·13	123·10·11·14·15	14589·10·11	145·12·13·14·15	D_5
2367	12389·14·15	123·10·11·12·13	14589·12·13	145·10·11·14·15	D_3
2389	1238·10·12·14	1239·11·13·15	24689·10·11	246·12·13·14·15	D_3
23·10·11	1238·11·12·15	1239·10·13·14	25789·10·11	257·12·13·14·15	D_3
23·12·13	1238·10·13·15	1239·11·12·14	34789·10·11	347·12·13·14·15	D_3
23·14·15	1238·11·13·14	1239·10·12·15	35689·10·11	356·12·13·14·15	D_3
248·14	12489·10·12	13589·11·13	2368·10·11·14	4568·12·13·14	D_3
249·15	12489·11·13	13589·10·12	2369·10·11·15	4569·12·13·15	D_3
24·10·12	12589·11·12	13489·10·13	2378·10·11·15	4578·12·13·15	D_3
24·11·13	12589·10·13	13489·11·12	2379·10·11·14	4579·12·13·14	D_3
258·15	1248·10·11·14	1359·10·11·15	23689·10·12	456·10·12·14·15	D_3
259·14	1249·10·11·15	1358·10·11·14	23689·11·13	456·11·13·14·15	D_3
25·10·13	1258·10·11·15	1349·10·11·14	23789·11·12	457·11·12·14·15	D_3
25·11·12	1259·10·11·14	1348·10·11·15	23789·10·13	457·10·13·14·15	D_3
348·15	1248·12·13·14	1359·12·13·15	236·10·12·14·15	45689·10·12	D_3
349·14	1249·12·13·15	1358·12·13·14	236·11·13·14·15	45689·11·13	D_3
34·10·13	1258·12·13·15	1349·12·13·14	237·11·12·14·15	45789·11·12	D_3
34·11·12	1259·12·13·14	1348·12·13·15	237·10·13·14·15	45789·10·13	D_3
358·14	124·10·12·14·15	135·11·13·14·15	2368·12·13·14	4568·10·11·14	D_3
359·15	124·11·13·14·15	135·10·12·14·15	2369·12·13·15	4569·10·11·15	D_3
35·10·12	125·11·12·14·15	134·10·13·14·15	2378·12·13·15	4578·10·11·15	D_3
35·11·13	125·10·13·14·15	134·11·12·14·15	2379·12·13·14	4579·10·11·14	D_3
4567	14589·14·15	145·10·11·12·13	16789·12·13	167·10·11·14·15	D_3
4589	1458·10·12·14	1459·11·13·15	24689·12·13	246·10·11·14·15	D_3
45·10·11	1458·11·12·15	1459·10·13·14	25789·12·13	257·10·11·14·15	D_3
45·12·13	1458·10·13·15	1459·11·12·14	34789·12·13	347·10·11·14·15	D_3
45·14·15	1458·11·13·14	1459·10·12·15	35689·12·13	356·10·11·14·15	D_3
6789	1678·10·12·14	1679·11·13·15	24689·14·15	246·10·11·12·13	D_3
67·10·11	1678·11·12·15	1679·10·13·14	25789·14·15	257·10·11·12·13	D_3
67·12·13	1678·10·13·15	1679·11·12·14	34789·14·15	347·10·11·12·13	D_3
67·14·15	1678·11·13·14	1679·10·12·15	35689·14·15	356·10·11·12·13	D_3
89·10·11	2468·11·12·15	2469·10·13·14	2578·10·12·14	2579·11·13·15	D_3
89·12·13	2468·10·13·15	2469·11·12·14	3478·10·12·14	3479·11·13·15	D_3
89·14·15	2468·11·13·14	2469·10·12·15	3568·10·12·14	3569·11·13·15	D_5
8·10·12·14	2478·10·12·15	2568·11·12·14	3468·10·13·14	3578·11·13·15	D_3
8·10·13·15	2478·11·12·14	2568·10·12·15	3469·10·12·14	3579·11·12·15	D_3
8·11·12·15	2478·10·13·14	2569·10·12·14	3468·10·12·15	3579·10·13·15	D_3
8·11·13·14	2479·10·12·14	2568·10·13·14	3468·11·12·14	3579·11·13·14	D_3
9·10·12·15	2478·11·13·15	2569·11·12·15	3469·10·13·15	3578·10·12·15	D_3
9·10·13·14	2479·11·12·15	2568·11·13·15	3469·11·13·14	3578·11·12·14	D_3
9·11·12·14	2479·10·13·15	2569·11·13·14	3468·11·13·15	3578·10·13·14	D_3
9·11·13·15	2479·11·13·14	2569·10·13·15	3469·11·12·15	3579·10·12·14	D_3
10·11·12·13	2578·10·13·15	2579·11·12·14	3478·11·12·15	3479·10·13·14	D_5
10·11·14·15	2578·11·13·14	2579·10·12·15	3568·11·12·15	3569·10·13·14	D_3
12·13·14·15	3478·11·13·14	3479·10·12·15	3568·10·13·15	3569·11·12·14	D_3

H-transformations of D_5

1247	1234589	12367·10·11	14567·12·13	189·10·11·12·13	D_4
1256	12345·10·11	1236789	14567·14·15	189·10·11·14·15	D_4
128·11	123468·10	123579·11	24567·12·14	289·10·11·12·14	D_4
129·10	123469·11	123578·10	24567·13·15	289·10·11·13·15	D_4
12·12·15	123478·11	123569·10	34567·12·15	389·10·11·12·15	D_4
12·13·14	123479·10	123568·11	34567·13·14	389·10·11·13·14	D_4
1346	12345·12·13	12367·14·15	1456789	189·12·13·14·15	D_4
1357	12345·14·15	12367·12·13	14567·10·11	1·10·11·12·13·14·15	D_4
138·10	12346·12·14	12357·13·15	245678·10	28·10·12·13·14·15	D_4
139·11	12346·13·15	12357·12·14	245679·11	29·11·12·13·14·15	D_4
13·12·14	12347·12·15	12356·13·14	345678·11	38·11·12·13·14·15	D_4
13·13·15	12347·13·14	12356·12·15	345679·10	39·10·12·13·14·15	D_4
148·13	124568·12	134579·13	23467·10·14	489·10·12·13·14	D_4
149·12	124569·13	134578·12	23467·11·15	489·11·12·13·15	D_4
14·10·15	124578·13	134569·12	23567·10·15	589·10·12·13·15	D_4
14·11·14	124579·12	134568·13	23567·11·14	589·11·12·13·14	D_4
158·12	12456·10·14	13457·11·15	234678·12	48·10·11·12·14·15	D_4
159·13	12456·11·15	13457·10·14	234679·13	49·10·11·13·14·15	D_4
15·10·14	12457·10·15	13456·11·14	235678·13	58·10·11·13·14·15	D_4
15·11·15	12457·11·14	13456·10·15	235679·12	59·10·11·12·14·15	D_4
168·15	124678·14	135679·15	23456·10·12	689·10·12·14·15	D_4
169·14	124679·15	135678·14	23456·11·13	689·11·13·14·15	D_4
16·10·13	125678·15	134679·14	23457·10·13	789·10·13·14·15	D_4
16·11·12	125679·14	134678·15	23457·11·12	789·11·12·14·15	D_4
178·14	12467·10·12	13567·11·13	234568·14	68·10·11·12·13·14	D_4
179·15	12467·11·13	13567·10·12	234569·15	69·10·11·12·13·15	D_4
17·10·12	12567·10·13	13467·11·12	234578·15	78·10·11·12·13·15	D_4
17·11·13	12567·11·12	13467·10·13	234579·14	79·10·11·12·13·14	D_4
2345	12389·12·13	123·10·11·14·15	14589·10·11	145·12·13·14·15	D_4
2367	12389·14·15	123·10·11·12·13	14589·12·13	145·10·11·14·15	D_4
2389	1238·10·12·14	1239·11·13·15	24689·10·11	246·12·13·14·15	D_4
23·10·11	1238·10·13·15	1239·11·12·14	25789·10·11	257·12·13·14·15	D_4
23·12·13	1238·11·12·15	1239·10·13·14	34789·10·11	347·12·13·14·15	D_4
23·14·15	1238·11·13·14	1239·10·12·15	35689·10·11	356·12·13·14·15	D_4
248·14	12489·10·12	13589·11·13	2368·10·11·14	4568·12·13·14	D_4
249·15	12489·11·13	13589·10·12	2369·10·11·15	4569·12·13·15	D_4
24·10·12	12589·10·13	13489·11·12	2378·10·11·15	4578·12·13·15	D_4
24·11·13	12589·11·12	13489·10·13	2379·10·11·14	4579·12·13·14	D_4
258·15	1248·10·11·14	1359·10·11·15	23689·10·12	456·10·12·14·15	D_4
259·14	1249·10·11·15	1358·10·11·14	23689·11·13	456·11·13·14·15	D_4
25·10·13	1258·10·11·15	1349·10·11·14	23789·10·13	457·10·13·14·15	D_4
25·11·12	1259·10·11·14	1348·10·11·15	23789·11·12	457·11·12·14·15	D_4
268·12	12689·10·14	13789·11·15	2348·10·11·12	4678·12·14·15	D_4
269·13	12689·11·15	13789·10·14	2349·10·11·13	4679·13·14·15	D_4
26·10·14	12789·10·15	13689·11·14	2358·10·11·13	5678·13·14·15	D_4
26·11·15	12789·11·14	13689·10·15	2359·10·11·12	5679·12·14·15	D_4
278·13	1268·10·11·12	1379·10·11·13	23489·10·14	467·10·12·13·14	D_4
279·12	1269·10·11·13	1378·10·11·12	23489·11·15	467·11·12·13·15	D_4
27·10·15	1278·10·11·13	1369·10·11·12	23589·10·15	567·10·12·13·15	D_4
27·11·14	1279·10·11·12	1368·10·11·13	23589·11·14	567·11·12·13·14	D_4
348·15	1248·12·13·14	1359·12·13·15	236·10·12·14·15	45689·10·12	D_4
349·14	1249·12·13·15	1358·12·13·14	236·11·13·14·15	45689·11·13	D_4
34·10·13	1258·12·13·15	1349·12·13·14	237·10·13·14·15	45789·10·13	D_4
34·11·12	1259·12·13·14	1348·12·13·15	237·11·12·14·15	45789·11·12	D_4
358·14	124·10·12·14·15	135·11·13·14·15	2368·12·13·14	4568·10·11·14	D_4
359·15	124·11·13·14·15	135·10·12·14·15	2369·12·13·15	4569·10·11·15	D_4
35·10·12	125·10·13·14·15	134·11·12·14·15	2378·12·13·15	4578·10·11·15	D_4
35·11·13	125·11·12·14·15	134·10·13·14·15	2379·12·13·14	4579·10·11·14	D_4
368·13	1268·12·14·15	1379·13·14·15	234·10·12·13·14	46789·10·14	D_4

369·12	1269·13·14·15	1378·12·14·15	234·11·12·13·15	46789·11·15	D_4
36·10·15	1278·13·14·15	1369·12·14·15	235·10·12·13·15	56789·10·15	D_4
36·11·14	1279·12·14·15	1368·13·14·15	235·11·12·13·14	56789·11·14	D_4
378·12	126·10·12·13·14	137·11·12·13·15	2348·12·14·15	4678·10·11·12	D_4
379·13	126·11·12·13·15	137·10·12·13·14	2349·13·14·15	4679·10·11·13	D_4
37·10·14	127·10·12·13·15	136·11·12·13·14	2358·13·14·15	5678·10·11·13	D_4
37·11·15	127·11·12·13·14	136·10·12·13·15	2359·12·14·15	5679·10·11·12	D_4
4567	14589·14·15	145·10·11·12·13	16789·12·13	167·10·11·14·15	D_4
4589	1458·10·12·14	1459·11·13·15	24689·12·13	246·10·11·14·15	D_4
45·10·11	1458·10·13·15	1459·11·12·14	25789·12·13	257·10·11·14·15	D_4
45·12·13	1458·11·12·15	1459·10·13·14	34789·12·13	347·10·11·14·15	D_4
45·14·15	1458·11·13·14	1459·10·12·15	35689·12·13	356·10·11·14·15	D_4
468·10	14689·12·14	15789·13·15	2458·10·12·13	2678·10·14·15	D_4
469·11	14689·13·15	15789·12·14	2459·11·12·13	2679·11·14·15	D_4
46·12·14	14789·12·15	15689·13·14	3458·11·12·13	3678·11·14·15	D_4
46·13·15	14789·13·14	15689·12·15	3459·10·12·13	3679·10·14·15	D_4
478·11	1468·10·12·13	1579·11·12·13	24589·12·14	267·10·11·12·14	D_4
479·10	1469·11·12·13	1578·10·12·13	24589·13·15	267·10·11·13·15	D_4
47·12·15	1478·11·12·13	1569·10·12·13	34589·12·15	367·10·11·12·15	D_4
47·13·14	1479·10·12·13	1568·11·12·13	34589·13·14	367·10·11·13·14	D_4
568·11	1468·10·14·15	1579·11·14·15	245·10·11·12·14	26789·12·14	D_4
569·10	1469·11·14·15	1578·10·14·15	245·10·11·13·15	26789·13·15	D_4
56·12·15	1478·11·14·15	1569·10·14·15	345·10·11·12·15	36789·12·15	D_4
56·13·14	1479·10·14·15	1568·11·14·15	345·10·11·13·14	36789·13·14	D_4
578·10	146·10·11·12·14	157·10·11·13·15	2458·10·14·15	2678·10·12·13	D_4
579·11	146·10·11·13·15	157·10·11·12·14	2459·11·14·15	2679·11·12·13	D_4
57·12·14	147·10·11·12·15	156·10·11·13·14	3458·11·14·15	3678·11·12·13	D_4
57·13·15	147·10·11·13·14	156·10·11·12·15	3459·10·14·15	3679·10·12·13	D_4
6789	1678·10·12·14	1679·11·13·15	24689·14·15	246·10·11·12·13	D_4
67·10·11	1678·10·13·15	1679·11·12·14	25789·14·15	257·10·11·12·13	D_4
67·12·13	1678·11·12·15	1679·10·13·14	34789·14·15	347·10·11·12·13	D_4
67·14·15	1678·11·13·14	1679·10·12·15	35689·14·15	356·10·11·12·13	D_4
89·10·11	2468·10·13·15	2469·11·12·14	2578·10·12·14	2579·11·13·15	D_4
89·12·13	2468·11·12·15	2469·10·13·14	3478·10·12·14	3479·11·13·15	D_4
89·14·15	2468·11·13·14	2469·10·12·15	3568·10·12·14	3569·11·13·15	D_4
8·10·12·14	2478·10·12·15	2568·10·13·14	3468·11·12·14	3578·11·13·15	D_4
8·10·13·15	2478·10·13·14	2568·10·12·15	3469·10·12·14	3579·10·13·15	D_4
8·11·12·15	2478·11·12·14	2569·10·12·14	3468·10·12·15	3579·11·12·15	D_4
8·11·13·14	2479·10·12·14	2568·11·12·14	3468·10·13·14	3579·11·13·14	D_4
9·10·12·15	2478·11·13·15	2569·10·13·15	3469·11·12·15	3578·10·12·15	D_4
9·10·13·14	2479·10·13·15	2568·11·13·15	3469·11·13·14	3578·10·13·14	D_4
9·11·12·14	2479·11·12·15	2569·11·13·14	3468·11·13·15	3578·11·12·14	D_4
9·11·13·15	2479·11·13·14	2569·11·12·15	3469·10·13·15	3579·10·12·14	D_4
10·11·12·13	2578·11·12·15	2579·10·13·14	3478·10·13·15	3479·11·12·14	D_4
10·11·14·15	2578·11·13·14	2579·10·12·15	3568·10·13·15	3569·11·12·14	D_4
12·13·14·15	3478·11·13·14	3479·10·12·15	3568·11·12·15	3569·10·13·14	D_4

therefore

$D_1 \overset{2}{\sim} 21D_3$, $D_2 \overset{2}{\sim} 21D_3$, $D_3 \overset{2}{\sim} 12D_1+12D_2+9D_4$, $D_4 \overset{2}{\sim} 54D_3+3D_5$, $D_5 \overset{2}{\sim} 105D_4$

and $\{D_1\}^2=\{D_1, D_2, D_3, D_4, D_5\}$

References

R. A. Fisher and F. Yates (1963) Statistical Tables for Biological, Agricultural and Medical Research. 6th ed. Hafner, New York.

Q. M. Hussain (1945b) On the totality of the solutions for the symmetrical incomplete block designs: $\lambda=2$, $k=5$ or 6. Sankhyā 7, 204-208.

S. Kageyama and A. Hedayat (1983) The family of t-designs -part II. J. Statist. Plann. Inference 7, 257-287.

T. Nakano, M. Endo and Y. Jitsukawa (1975) Arithmetic in block designs II. Comment Math. Univ. St. Pauli XXIV-1, 27-46.

D. A. Preece (1967) Incomplete block designs with v=2k. Sankhyā Ser. A 29, 305-316.

K. Takeuchi (1962) A table of difference sets generating balanced incomplete block designs. Rev. Inst. Internat. Statist. 30, 361-366.

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