On solvability of a time-fractional semilinear heat equation, and its quantitative approach to the classical counterpart.

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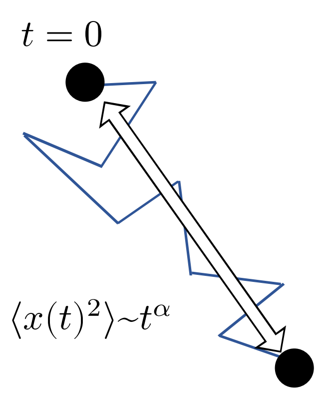
1. Introduction

	Fujita equation	Time-fractional equation
	(F) $\begin{cases}\partial_t u - \Delta u = u^p & \text{in } (0,T)\times\mathbb{R}^N \\ u(0)=u_0 & \text{on } \mathbb{R}^N\end{cases}$	$(\mathrm{F})_\alpha$ $\begin{cases}\partial_t^\alpha u - \Delta u = u^p & \text{in } (0,T)\times\mathbb{R}^N \\ u(0)=u_0 & \text{on } \mathbb{R}^N\end{cases}$
Global-in-time solvability	• $1 < p \le p_F := 1 + \frac{2}{N} \Rightarrow$ No global sol. • $p_F < p \Rightarrow \exists$ Global sol.	• $1 < p < p_F \Rightarrow$ No global sol. • $p_F \le p \Rightarrow \exists$ Global sol.
Local-in-time solvability	For $p = p_F, \exists\, u_0 \in L^1(\mathbb{R}^N)$ s.t. (F) has no sol. (doubly critical)	For $p = p_F, \forall u_0 \in L^1(\mathbb{R}^N)$, $(\mathrm{F})_\alpha$ has a local-in-time sol.



$$(\partial_t^\alpha f)(t) := \frac{1}{\Gamma(1-\alpha)}\frac{\partial}{\partial t}\int_0^t (t-\tau)^{-\alpha}\left(f(\tau)-f(0)\right)d\tau, \quad 0<\alpha<1$$

Caputo derivative of order $\alpha \in (0,1)$.

Remark
(F) and $(\mathrm{F})_\alpha$ are scale invariant in $L^1(\mathbb{R}^N)$ if $p = p_F$.

See e.g. [Zhang, Sun '15], [Ghergu, Miyamoto, Suzuki '22]

<u>Subdiffusion process</u> :Diffusion in heterogeneous media, contaminants in Soil, etc.
c.f.
- $\langle x(t)^2\rangle \sim t \Rightarrow$ Brownian motion; $\partial_t u - \Delta u = 0$.
- $\langle x(t)^\beta\rangle \sim t,\ 0<\beta<2 \Rightarrow$ Superdiffusion; $\partial_t u + (-\Delta)^{\beta/2} u = 0$.

Question
Does $(\mathrm{F})_\alpha$ with $p = p_F$ approach (F) when $\alpha \to 1$? And how?

2. Necessary and sufficient condition [Hisa, K, arXiv]

N.C. Suppose that $(\mathrm{F})_\alpha$ has a solution on $(0,T)$. Then, $\exists C(\alpha) > 0$ s.t.

$$\sup_{z\in\mathbb{R}^N}\int_{B(z;\rho)} u_0(y)dy \le C(\alpha)\left[\int_{\rho^{2/\alpha}/(16\ \ )}^{1/4} t^{-\alpha}\,dt\right]^{-N/2}$$

for $0 < \rho^{2/\alpha} < T$. Moreover, we have $\limsup_{\alpha\to 1} C(\alpha) < \infty$.

S.C. $\exists c > 0$ s.t. if

$$\sup_{z\in\mathbb{R}^N}\int_{B(z;T^{\alpha/2})} u_0(y)dy \le c(1-\alpha)^{N/2},$$

then, $(\mathrm{F})_\alpha$ has a solution on $(0,T)$.

See e.g.
[Baras, Pierre '85], [Hisa, Ishige '18] for necessary condition;
[Oka, Zhanpeisov, arXiv] for sufficient condition.

3. Main results [Hisa, K, arXiv]

Thm.1 (Global solvability) Let

$$G_\alpha := \{v \in L^1(\mathbb{R}^N);\ v \ge 0 \text{ and } (F)_\alpha \text{ with } u_0 = v \text{ has a global sol.}\}$$

Then, $\exists C > c > 0$ s.t.

$$c(1-\alpha)^{N/2} \le \sup_{v\in G_\alpha}\|v\|_{L^1(\mathbb{R}^N)} \le C\,(1-\alpha)^{N/2}$$

near $\alpha = 1$.

Thm.2 (Local solvability) Let

$\mu_\epsilon(x) = |x|^{-N}\left(-\log|x|\right)^{-N/2\,-\,1+\epsilon}\chi_{B(e^{-1})}$ and

$$T_\alpha := \sup\{T>0; (F)_\alpha \text{ with } u_0 = \mu_\epsilon \text{ has a sol. on } (0,T).\}$$

Then, $\exists C > c > 0$ s.t.

$$\exp\left(-C(1-\alpha)^{-\frac{N}{N-2\epsilon}}\right) \le T_\alpha \le \exp\left(-c(1-\alpha)^{-\frac{N}{N-2\epsilon}}\right).$$

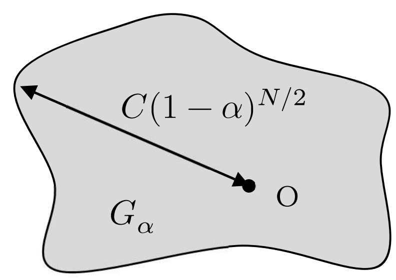


Figure
As $\alpha \to 1$, the initial value must be infinitely small to obtain a global solution.

- Thm.1 and Thm.2 represent the *collapse* of the global and local solvability for the time-fractional equation.
- For the proof of theorems, we use
  Under estimate⋯Sufficient condition
  Upper estimate⋯Necessary condition

(F) with $u_0 = \mu_\epsilon$ admits no local-in-time solutions.

See [Hisa, Ishige '15], [Miyamoto '21]