Virtual physical reservoir computing with mathematical models

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1 Introduction

Physical systems can be reinterpreted as powerful computational resources through the lens of reservoir computing [1, 2, 3], an information processing technique that exploits the input-driven transient behaviors of high-dimensional dynamical systems. Focusing on the intrinsic information processing capabilities of such systems, we introduce two "virtual" settings of physical reservoir computing.

2 Flow past a circular cylinder as a reservoir [4]

The first system is the flow past a circular cylinder governed by the two-dimensional incompressible Navier–Stokes equations:

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} - \nabla\cdot\sigma(\mathbf{v},p) = \mathbf{0}, \quad \nabla\cdot\mathbf{v} = 0 \qquad \text{in } \Omega\times(0,T_f), \tag{1a}$$

$$\mathbf{v} = \mathbf{u}_* \qquad \text{on } \Gamma_1\times(0,T_f), \tag{1b}$$

$$[\sigma(\mathbf{v},p)\mathbf{n}]\cdot\mathbf{n}^\perp = 0,\ \mathbf{v}\cdot\mathbf{n} = 0 \qquad \text{on } \Gamma_2\times(0,T_f), \tag{1c}$$

$$\sigma(\mathbf{v},p)\mathbf{n} = \mathbf{0} \qquad \text{on } \Gamma_3\times(0,T_f), \tag{1d}$$

$$\mathbf{v} = \mathbf{0} \qquad \text{on } \Gamma_4\times(0,T_f), \tag{1e}$$

with an initial velocity, where Ω and $(0,T_f)$ are domains in space and time for $T_f > 0$, respectively, $\mathbf{v} = (v_1,v_2)^\top : \Omega\times(0,T_f) \to \mathbb{R}^2$ is the velocity, $p\colon \Omega\times(0,T_f)\to\mathbb{R}$ is the pressure, $\sigma(\mathbf{v},p) \coloneqq (2/\mathrm{Re})D(\mathbf{v}) - pI : \Omega\times(0,T_f)\to\mathbb{R}^{2\times2}$ is the stress tensor for the Reynolds number $\mathrm{Re} > 0$, $D(\mathbf{v}) \coloneqq (1/2)[\nabla\mathbf{v} + (\nabla\mathbf{v})^\top] : \Omega\times(0,T_f)\to\mathbb{R}^{2\times2}$ is the strain-rate tensor, $I\in\mathbb{R}^{2\times2}$ is the identity tensor, $\mathbf{n} : \partial\Omega\to\mathbb{R}^2$ is the outward unit normal vector, $\mathbf{n}^\perp : \partial\Omega\to\mathbb{R}^2$ is the unit tangential vector, and $\mathbf{u}_* : \Gamma_1\times(0,T_f)\to\mathbb{R}^2$ is a given inflow boundary velocity. We set $\mathbf{u}_* = (u,0)^\top$ for an input function $u = u(t)\in\mathbb{R}$; the input u takes a random value at each fixed time interval, and the range of u is $[0.99, 1.01]$. We employ the velocity and pressure values at 77 points behind the cylinder as the state variables of the reservoir. Changing the Reynolds number, we observe the information processing capability of the reservoir. As a result, it is shown that the computational capability is the highest near $\mathrm{Re} = 45$, which is just before the onset of a Kármán vortex street and corresponds to the twin vortex.

3 Kuramoto model as a reservoir [5]

The second system is the Kuramoto model [6], a ubiquitous model for phase-locking—specifically, the version with a forcing term for synchronization to an external input [7, 8]. Let $u(t) = (u_1(t), u_2(t), \dots, u_M(t)) \in \mathbb{R}^M$, $M\in\mathbb{N}$, be an input. The Kuramoto model with the forcing term is given by

$$\frac{d\theta_k}{dt} = \omega_k + \frac{K}{N}\sum_{j=1}^{N}[\sin(\theta_j - \theta_k)] + F\sin(cu_{v_k} - \theta_k), \tag{2}$$

for $k = 1, \ldots, N$, where ω_k are sampled from $2\pi\mathcal{N}$ with $\mathcal{N}$ denoting the normal distribution with mean 1 and standard deviation 1, v_k are uniformly sampled from $\{1, 2, \ldots, M\}$, and c is the input scaling constant. The parameters $K, F \geq 0$ are referred to as the coupling and forcing parameters, respectively. Here the reservoir is given by coupled Kuramoto oscillators subject to a forcing term driven by the input. Then, by decoupling the system from the input, substituting the input with a feedback loop, the network can reproduce the attractor autonomously. This input-decoupled system is commonly referred to as the closed system. We obtained the following results. The closed system is successful when a self-sustained synchronization is achieved. It has a wide range of successful hyperparameter configurations. Additionally, for hyperparameters that lead to failure, we present a simple test to identify oscillators that fail to reproduce the attractor's geometry, i.e., the rotation number test.

4 Conclusion

We have introduced two settings of virtual physical reservoir computing with mathematical models. Further details will be presented in the talk.

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Stochastic Singular Control Problems in Utility Maximization

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1 Introduction

Recent studies have explored a wide range of consumption constraints in utility maximization, including habit formation, ratcheting, drawdown limits, and subsistence constraints. However, relatively little attention has been paid to the endogenous adjustment of consumption bounds. While [1] considered fixed bounds, [2] introduced a preference model where the lower bound of consumption can be dynamically updated, though their analysis was limited to an infinite-horizon setting.

This study investigates the optimal consumption and investment problem for a finitely-lived agent with constant relative risk aversion (CRRA) utility, whose consumption bound can be endogenously adjusted at a cost. The agent derives utility from both consumption and the minimum level of past consumption, which determines the consumption bound. Adjustments incur a proportional utility cost, and thus the minimum level evolves as a finite variation process. The agent chooses consumption, portfolio, and minimum consumption level over a finite horizon, leading to a stochastic singular control problem with state constraints.

To solve this problem, we develop a duality approach. First, we derive the dual singular control problem using the dual-martingale method. Then, using the connection between singular and switching controls ([3]), we reformulate the dual as a one-dimensional switching problem. Finally, we characterize the value function via a parabolic double obstacle problem and rigorously analyze the solution' s existence, uniqueness, and regularity using PDE theory. We construct optimal switching strategies and recover the optimal singular control as a Skorokhod problem with two free boundaries.

Our results show that the agent' s optimal strategy is governed by the wealth-to-consumption ratio. Adjustments occur when this ratio hits time-dependent thresholds. The risky asset share is always less than or equal to the Merton share and coincides with it upon adjustment. When adjustment costs are positive, there exists a critical time beyond which the minimum consumption level is never increased.

A main technical contribution lies in our analysis of the double obstacle problem, which involves two non-monotonic free boundaries. Existing literature on such problems typically assumes monotonicity for regularity results (e.g., [4], [5]). We extend the method of [6] by applying a twisted domain and advanced PDE estimates to establish smoothness despite irregular

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nonhomogeneous terms.

Our framework generalizes prior models involving fixed bounds ([1]), drawdown constraints ([7], [8]), and costly consumption adjustment ([9], [10]).

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From Solid Tumors to Liquid Condensates: A Phase–Field, Mechano–Chemical View

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1 Introduction Many pathophysiological phenomena entail *multiphase, multiscale* material re-organisation driven by mechanics and chemistry. Two apparently disparate examples are (i) the expansion of a solid tumour in living tissue and (ii) the formation of membraneless biomolecular condensates (“liquid droplets”) inside cells. Both systems exhibit sharp or diffuse phase boundaries, interfacial stresses and chemical reactions. By adopting phase-field framework, we show how ideas and numerics applied in both tumour modelling and intracellular condensate dynamics.

2 Mechano-Chemical Model of Tumour Growth We adopt an energetic variational approach in which (i) mechanical work, (ii) interfacial free energy, and (iii) reaction-diffusion of nutrients and extracellular matrix components jointly determine evolution. Let $\phi(\mathbf{x},t)$ be the tumour phase-field ($\phi=1$: tumour, $\phi=0$: host). The total free energy

$$\mathcal{F}=\int_\Omega \Big\{\gamma\big[\tfrac{\epsilon^2}{2}|\nabla\phi|^2+f(\phi)\big]+\tfrac{k}{2}c^2+J_e^{-1}W_e(\phi,F_e)\Big\}dx$$

combines Cahn–Hilliard interfacial energy (γ), elastic energy W_e, and chemical free-energy density (k). The coupled governing equations read

$$\begin{aligned}
\phi_t+\nabla\cdot(\phi\mathbf{u}) &= \nabla\cdot\big[M(\phi)\nabla(\gamma\mu)\big]+S_\phi,\\
c_t+\nabla\cdot(c\mathbf{u}) &= D\Delta c-S_c,\\
\nabla\cdot\mathbf{u}=S_\phi,\qquad & \alpha\mathbf{u}=\nabla\cdot\sigma+\gamma\mu\nabla\phi,\\
\partial_t\mathbf{Y}+\mathbf{u}\cdot\nabla\mathbf{Y} &= 0.
\end{aligned}$$

Numerical simulations in both sharp interface and diffuse interface formulation will be given.

3 Phase-Field View of Biomolecular Condensates The preprint [1] looks into intracellular liquid–liquid phase separation (LLPS). Using an energy-stable scalar auxiliary variable (SAV) scheme to time-integrate the Cahn–Hilliard (CH) equation, we built robust Python/MATLAB/Julia solvers that preserve the free-energy dissipation law at every step; this numerical backbone let us simulate, sans artefacts, the entire life cycle of biomolecular condensates—nucleation, growth, dewetting, and coarsening. By quantifying how the SAV-driven CH dynamics tie critical droplet radius to the diffuse-interface coefficient, we uncovered a size threshold that we then mapped onto the chromosomal passenger complex (CPC): droplets predicted below the threshold dissolve, whereas larger ones persist and coarsen exactly as seen in live-cell experiments across multiple cell types. Thus, the SAV-enhanced CH solver not only ensures mathematical stability but also translates directly into mechanistic insight—verifying that CPC behaves as a phase-separated liquid and offering a generalizable framework for probing condensate biology.

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