分数べき Fourier 変換による Paley-Wiener の定理とサンプリング定理

Paley-Wiener Theorem and Sampling Theorem by Fractional Fourier Transform

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1　はじめに

分数べき Fourier 変換は Fourier 変換の一般化として，様々な分野で応用されている．$i=\sqrt{-1}$ として，分数べきフーリエ変換は積分核

$$K_\alpha(x,y)=\begin{cases}\dfrac{1}{\sqrt{\pi(1-e^{-2\alpha i})}}\exp\left(i\dfrac{(x^2+y^2)\cos\alpha-2xy}{2\sin\alpha}\right) & (\alpha\in\mathbf{R}/\pi\mathbf{Z})\\ \delta(x-y) & (\alpha\in 2\pi\mathbf{Z})\\ \delta(x+y) & (\alpha+\pi\in 2\pi\mathbf{Z})\end{cases}$$

を持つ積分変換

$$\mathcal{F}_\alpha(f)(y)=\int_{\mathbf{R}} f(x)K_\alpha(x,y)\ dx,$$

として定義される（[1], [2] 参照）．ここで，δ は Dirac のデルタ関数を表す．特に分数べき Fourier 変換は，$\alpha=\pi/2$ のときは Fourier 変換，$\alpha=-\pi/2$ のときは Fourier 逆変換と一致することが知られている．

Paley-Wiener の定理は Fourier 変換によって，関数の台とその解析的性質との関係を与える定理である．特に台がコンパクトである関数の Fourier 変換の像は，複素平面上のある種の増大度をもつ整関数となることが知られている．

定理 1　$A>0$，$S_A=[-A,A]$ とし，U を $\mathbf{C}$ 上の整関数とする．整関数 U が，S_A に台をもつ関数 $u\in C_0^\infty(\mathbf{R})$ の Fourier 変換の像として表すことができるための必要十分条件は，任意の $L\in\mathbf{Z}_+$ に対してある定数 $C>0$ が存在し，

$$|U(\zeta)|\leq C(1+|\zeta|)^{-L}\exp\left(A|\operatorname{Im}\zeta|\right),\quad \zeta\in\mathbf{C}$$

を満たすことである．

さらに U が S_A に台を持つ超関数 $u\in\mathcal{E}'(\mathbf{R})$ の Fourier-Laplace 変換の像として表すことができるための必要十分条件は，ある定数 $C>0$ と $L\in\mathbf{Z}_+$ が存在して，

$$|U(\zeta)|\leq C(1+|\zeta|)^{L}\exp\left(A|\operatorname{Im}\zeta|\right),\quad \zeta\in\mathbf{C}$$

が成り立つことである．

この定理は数学の各分野だけにとどまらず，サンプリング定理など応用においても重要な役割を果たしている．本研究ではこの Paley-Wiener の定理を分数べき Fourier 変換へと拡張する．

2　主結果

本研究では定理 1 を一般化した．まずは Paley-Wiener の定理を分数べき Fourier 変換へと一般化した主張を述べる．

定理 2　U を $\mathbf{C}$ 上の整関数とする．このとき，U は S_A 内に台をもつ関数 $u \in C_0^\infty(\mathbf{R})$ のパラメータ $\alpha \in (0,\pi)$ の分数べき Fourier 変換の像として表すことができるための必要十分条件は，任意の $L \in \mathbf{Z}_+$ に対して，ある定数 $C > 0$ が存在し，任意の $\zeta \in \mathbf{C}$ に対して，

$$|U(\zeta)| \le C(1+|\zeta|)^{-L} \exp\left(-\frac{\operatorname{Re}\zeta \operatorname{Im}\zeta}{\tan\alpha} + \frac{A|\operatorname{Im}\zeta|}{\sin\alpha}\right)$$

を満たすことである．

また，この定理を超関数まで拡張すると，次のような主張を得る．

定理 3　U を $\mathbf{C}$ 上の整関数とする．このとき，U は S_A 内に台をもつ超関数 $u \in \mathcal{E}'(\mathbf{R})$ のパラメータ $\alpha \in (0,\pi)$ の分数べき Fourier-Laplace 変換の像として表すことができるための必要十分条件は，ある定数 $C > 0$ と $L \in \mathbf{Z}_+$ が存在して，任意の $\zeta \in \mathbf{C}$ に対して，

$$|U(\zeta)| \le C(1+|\zeta|)^{L} \exp\left(-\frac{\operatorname{Re}\zeta \ \operatorname{Im}\zeta}{\tan\alpha} + \frac{A|\operatorname{Im}\zeta|}{\sin\alpha}\right)$$

を満たすことである．

3　サンプリング定理への応用

サンプリング定理は，関数を sinc 関数を用いて展開するための条件を与える定理である．[3] では分数べき Fourier 変換に基づいて，サンプリング定理を一般化している．本研究ではこの [3] の結果と我々の結果を結びつけた，次の主張を得ることができた．

定理 4　$\alpha \in (0,\pi)$, U を $\mathbf{C}$ 上の整関数とし，$L \in \mathbf{Z}_+, C > 0$ が存在して，任意の $\zeta \in \mathbf{C}$ に対して，

$$|U(\zeta)| \le C(1+|\zeta|)^{-L} \exp\left(-\frac{\operatorname{Re}\zeta \operatorname{Im}\zeta}{\tan\alpha} + \frac{A|\operatorname{Im}\zeta|}{\sin\alpha}\right)$$

を満たしているとする．このとき，U は

$$U(\zeta) = \exp\left(\frac{i\zeta^2}{2\tan\alpha}\right) \sum_{n\in\mathbf{Z}} U\left(\frac{n\pi}{A}\sin\alpha\right) \exp\left(-\frac{i\pi^2 n^2 \sin^2\alpha}{2A^2\tan\alpha}\right) \operatorname{sinc}\left(\frac{A\zeta}{\sin\alpha} - n\right) \tag{1}$$

という展開が可能である．ここで，(1) は $L^2(\mathbf{R})$ の意味で，また $\mathbf{C}$ 上の任意のコンパクト集合上で一様に収束する．

参考文献

[1] V. Namias : “The fractional order Fourier transform and its application to quantum mechanics”, J. Inst. Math. Its Appl., 25 (1980), pp. 241-265

[2] H. M. Ozaktas, Z. Zalevsky and M. A. Kutay : “The Fractional Fourier Transform with Applications in optics and Signal Processing”, Wiley 2001.

[3] Ç. Candan, H. M. Ozaktas : “Sampling and series expansion theorems for fractional Fourier and other transforms”, Signal Processing, **83**(11)(2003), 2455-2457.

古典的数学手法による地球物理データ中の時間変化する空間構造の探究

Exploring Time-Varying Spatial Structures in Geophysical Data with Classical Mathematical Tools

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1　概要

In this presentation, we introduce a mathematical analysis of the spatial structure of precipitation, using 44 years of monthly precipitation data from 53 observation stations provided by the Korea Meteorological Administration. While basic statistics can provide information about selected locations or specific time periods, they tend to reduce either spatial or temporal characteristics in the process. In particular, the spatial structure of precipitation is initially unknown, and measures such as the mean or variance often compress all the structural information. Therefore, a mathematical approach is required to explore the latent structure. A representative method for such exploration is Empirical Orthogonal Function (EOF) analysis, which is based on concepts from linear algebra, such as basis, inner product, and eigenvectors, and is implemented through singular value decomposition (SVD), a result of the spectral theorem. EOF analysis decomposes the precipitation data into signal and noise, allowing for a low-dimensional summary of spatial variability. In practice, the top three spatial modes reflect structures such as east-west asymmetry, north-south symmetry, and features influenced by latitude and topography. However, EOF provides fixed spatial patterns, failing to reflect the seasonal variability of precipitation throughout the year. To address this limitation, this presentation introduces Cyclo-Stationary EOF (CSEOF), an extended form of EOF. CSEOF embeds temporal periodicity into the spatial modes, allowing for the extraction of spatial structures that evolve over time. This enables a more dynamic interpretation of spatial forms throughout the temporal cycle. This presentation aims to reflect on how the shared mathematical foundation of EOF and CSEOF leads to different interpretations of the same dataset, and to explore how the perspective of spatio-temporal analysis can be expanded through these contrasting approaches.

　

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参考文献

[1] H. Bjornsson and S. A. Venegas, A manual for EOF and SVD analyses of climatic data, Canadian Climate Centre, 1997.

[2] D. L. Hartmann, ATM 552 Lecture Notes, University of Washington, 2006.

[3] A. Hannachi, I. T. Jolliffe and D. B. Stephenson, Empirical orthogonal functions and related techniques in atmospheric science: A review, International Journal of Climatology, 27(9) (2007), 1119-1152.

[4] A. Hannachi, Patterns Identification and Data Mining in Weather and Climate, Springer, 2021.

[5] H. Jeon and S. Kim, Introduction to cyclo-stationary EOF analysis, East Asian Mathematical Society, 41(2) (2025), 1-10.

[6] J. Wang, C.-S. Chern and A. K. Liu, The wavelet empirical orthogonal function and its application to the analysis of internal tides, Journal of Atmospheric and Oceanic Technology, 17(10) (2000), 1403-1416.

[7] H.-L. Yu and Y.-C. Lin, Analysis of space-time non-stationary patterns of rainfall-groundwater interactions by integrating empirical orthogonal function and cross wavelet transform methods, Journal of Hydrology, 525 (2015), 585-597.

[8] Y. R. Rao and C. R. Murthy, Comparative analysis of spectral, empirical orthogonal function, and wavelet decompositions for an upwelling event, Journal of Atmospheric and Oceanic Technology, 18(4) (2001), 704-710.

[9] E. Cho and C. Yoo, Analysis of the spatiotemporal behavior of precipitation in South Korea based on EOF and CSEOF analyses, Journal of the Korean Society of Hazard Mitigation, 20(4) (2020), 21-32.

[10] M. Sun, G. Kim, K. Lei and Y. Wang, Evaluation of technology for the analysis and forecasting of precipitation using cyclostationary EOF and regression method, Atmosphere, 13(3) (2022), 500.

地球科学における古典的手法を超えて：ウェーブレット比較による時空間固有関数フレームワーク

Beyond Classical Tools in Geoscience : A Spatio-Temporal Eigenfunction Framework via Wavelet Comparison

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1 概要

This presentation investigates the temporal dynamics embedded in precipitation data, which are inherently nonlinear and nonstationary, through a comparative analysis of two spatio-temporal methods: a two-stage approach combining Empirical Orthogonal Function (EOF) with wavelet decomposition, and a one-stage approach using Cyclo-Stationary EOF (CSEOF). The goal is to construct a mathematically grounded framework capable of capturing multi-scale temporal behavior and offering deeper insights into complex geophysical signals. The analysis begins by extracting dominant spatial modes via EOF, followed by wavelet decomposition of the associated time coefficients. This process enables time-frequency analysis of the principal components, allowing localized, scale-dependent features to emerge. By systematically comparing wavelet filters and decomposition levels, we identify configurations that minimize reconstruction error while preserving interpretability. The resulting EOF-Wavelet approach effectively uncovers nonlinear periodic structures that are obscured in conventional linear analysis. In contrast, CSEOF analysis directly models temporally evolving spatial patterns and their corresponding periodic time series in a unified step. This enables the capture of both spatial and temporal variability without decoupling the two dimensions. A detailed comparison of both methods in the time-frequency domain—using log-scaled power spectral analysis—reveals a recurring descent structure around 0.06 Hz, a potential signature of underlying nonlinear oscillatory behavior. CSEOF demonstrates superior capacity in representing this feature, supported by lower loss metrics and phase coherence. This presentation emphasizes not only the contrast between sequential and integrated modeling strategies but also the importance of choosing appropriate mathematical tools to analyze temporally complex environmental data. By comparing EOF-Wavelet and CSEOF, we propose a versatile framework for interpreting nonstationary spatio-temporal processes in the geosciences.

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参考文献

[1] H. Jeon and S. Kim, Introduction to cyclo-stationary EOF analysis, East Asian Mathematical Society, 41(2) (2025), 1-10.

[2] H. Jeon and S. Kim, Comparative analysis of EOF-wavelet and CSEOF methods for signal decomposition, Journal of the Korean Society for Industrial and Applied Mathematics (J. KSIAM), 13(4) (2025), 21-23.

[3] K. Y. Kim and Q. Wu, A comparison study of EOF techniques: Analysis of nonstationary data with periodic statistics, Texas A&M University Technical Report, 1999.

[4] K. Y. Kim, Cyclostationary EOF Analysis, Seoul National University Press, 2017.

[5] E. Cho and C. Yoo, Analysis of the spatiotemporal behavior of precipitation in South Korea based on EOF and CSEOF analyses, Journal of the Korean Society of Hazard Mitigation, 20(4) (2020), 21-32.

[6] M. Sun, G. Kim, K. Lei and Y. Wang, Evaluation of technology for the analysis and forecasting of precipitation using cyclostationary EOF and regression method, Atmosphere, 13(3) (2022), 500.

[7] K. Y. Kim, B. Hamlington and H. Na, Theoretical foundation of cyclostationary EOF analysis for geophysical and climatic variables: Concepts and examples, Earth-Science Reviews, 150 (2015), 201-218.

[8] H.-E. Jeon, K.-J. Ha and H.-R. Kim, A study on characteristics of climate variability and changes in weather indexes in Busan since 1904, Atmosphere (Korean Meteorological Society), 33(1) (2023), 1-20.

[9] J. C. Goswami and A. K. Chan, Fundamentals of Wavelets: Theory, Algorithms, and Applications, Wiley-Interscience, 1999.

[10] M. G. Cui, D. M. Lee and J. G. Lee, Fourier Transform and Wavelet Analysis, World Scientific Publishing, 2001.